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ESF
Exploratory Workshop

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With 102 Figures

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Preface

This volume contains revised papers that were presented at the international workshop entitled *Computational Methods for Algebraic Spline Surfaces* ("COMPASS"), which was held from September 29 to October 3, 2003, at Schloß Weinberg, Kefermarkt (Austria).

The workshop was mainly devoted to approximate algebraic geometry and its applications. The organizers wanted to emphasize the novel idea of approximate implicitization, that has strengthened the existing link between CAD / CAGD (Computer Aided Geometric Design) and classical algebraic geometry. The existing methods for exact implicitization (i.e., for conversion from the parametric to an implicit representation of a curve or surface) require exact arithmetic and are too slow and too expensive for industrial use. Thus the duality of an implicit representation and a parametric representation is only used for low degree algebraic surfaces such as planes, spheres, cylinders, cones and toroidal surfaces. On the other hand, this duality is a very useful tool for developing efficient algorithms. Approximate implicitization makes this duality available for *general* curves and surfaces.

The traditional exact implicitization of parametric surfaces produce global representations, which are exact everywhere. The surface patches used in CAD, however, are always defined within a small box only; they are obtained for a bounded parameter domain (typically a rectangle, or – in the case of "trimmed" surface patches – a subset of a rectangle). Consequently, a globally exact representation is not really needed in practice. Instead of a single exact high–degree implicit representation, the methods of approximate implicitization produce piecewise implicit surfaces of relatively low degree, which may cover the shape with any desired accuracy. This results in so–called *algebraic spline surfaces*, which can be expected to replace the exact implicit representation in many algorithms.

Compared to the traditional parametric representations, such as rational curves and surfaces (so–called Non-Uniform-Rational-B-Splines – NURBS), algebraic spline surfaces offer several computational advantages. For instance, by exploiting the duality between implicit and parametric representation, the intersection of two surfaces can easily be traced if one of the surfaces is given in implicit, and the other surface is given in parametric form. In this case, the problem can be reduced to a two-dimensional root-finding problem. In the case of two parametric surfaces, one has to solve a four–dimensional problem instead. As another advantage, the fitting of surfaces to scattered data, which is a fundamental tool for generating free-form geometry from prototypes, can be done without mapping the data into a plane – a process which often limits the flexibility and usefulness of the surface fitting techniques which are available today. We also foresee a number of other applications, e.g., in the computer game industry, virtual reality, medical imaging, and scientific computing.

The workshop, and the papers collected in this volume, was devoted both to the theoretical fundamentals and to the various computational aspects which arise in applications of approximate algebraic geometry. These applications are based on techniques developed in different branches of mathematics and computer science, including numerical analysis and scientific computing, algebraic geometry, applied geometry, and computer graphics. For instance, numerical methods are needed to efficiently generate implicit representations, and algebraic techniques are essential for detecting and analyzing singularities, which may help to solve practical problems arising in applied geometry and computer graphics.

Traditionally, these fields are represented by several fairly disjoint scientific communities, which traditionally do not communicate much. In order to stimulate the exchange of ideas, and to promote interdisciplinary research, the workshop brought together experts from the various fields involved.

The papers included in this volume provide an overview about the state-of-the-art in approximative implicitization and various related topics, including both the theoretical basis and the existing computational techniques. This can be expected to encourage and promote the use of approximate implicitization for solving geometric problems in computer-aided design. In some of the papers and in the panel discussion at COMPASS, which is also documented in this volume, the authors try to identify a number of problems (both theoretical and practical ones) which need to be addressed by the different research communities, in order to exploit the potential of implicit representations.

The editors are convinced that this volume will support the mutual exchange of ideas between the various research communities, promoting interdisciplinary research. The interactions between different mathematical disciplines such as approximation theory, classical algebraic geometry and computer aided geometric design will play an essential role for exploiting the new idea of approximate algebraic geometry.

The editors of this volume are indebted to the European Science Foundation (ESF) for providing generous financial support of the COMPASS event, which was organized as an ESF Exploratory workshop (EW 02/55). They would like to thank the staff at Springer–Verlag, Heidelberg, for the constructive cooperation during the preparation production of these proceedings. The time and the effort of the 37 referees, whose reports have greatly helped to improve the quality and the presentation of the material, is gratefully acknowledged. Last, but not least, they would like to thank Elmar Wurm and Martin Aigner for collecting the papers and preparing the final manuscript.

Oslo and Linz, May 2004

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Approximate Parametrisation of Confidence Sets*

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Abstract. In various geometrical applications, the analysis and the visualization of the error of calculated or constructed results is required. This error has very often character of a nontrivial multidimensional probability distribution. Such distributions can be represented in a geometrically interesting way by a system of so called confidence sets. In our paper we present a method for an approximate parametrisation of these sets. In sect. 1 we describe our motivation, which consists in the study of the errors of so called Passive Observation Systems (POS). In sect. 2 we give a result about the intersection of quadric surfaces of revolution, which is useful in the investigation of the POS. In sect. 3 we give a general method for an approximate parametrisation of the confidence sets via simultaneous Taylor expansion. This method, which can be applied in a wide range of geometrical situations, is demonstrated on a concrete example of the POS.

1 Motivation

Our research was motivated by concrete problem of the analysis and the visualization of the errors of so called Passive Observation Systems (POS).

1.1 Passive Observation Systems

The POS have been successfully constructed and produced in Czech Republic since the 1960's as an alternative to the classical radars. These systems, which do not transmit any signal (therefore passive), are based on the principle of the time difference. A pulse in the transmission of an object (a plane) is received at four (or more) observation sites. In practice any plane is forced to transmit some signals, at least in order to ensure its orientation. From the differences of the time of reception of the pulse the position of the object can be determined.

The POS have two main advantages comparing to the standard radars. As they do not transmit any signal they can not be itself detected and have very low energy consumption.

In addition the error of the POS has a different characteristic comparing to the classical radars. For this reason a simultaneous use of the POS and the classical radars can be very interesting. For more details about the principle of the POS and for the basic information about their precision see [1, Chapter 5].

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1.2 Geometry of POS

The construction of POS creates many difficult problems on the level of the electrical engineering, but the underlying geometry is quite simple. Let a pulse transmitted by an object X be received at the sites A and A' respectively at times t_A and $t_{A'}$. Multiplying the difference $t_A - t_{A'}$ by the speed of the signal (typically the speed of light) we get the difference $d_{AA'}$ of distances from the object X to the sites A and A'. The object X must therefore lie on one of sheets of the two-sheet hyperboloid of revolution, which is determined by its foci A, A' and the measured difference of distances $d_{AA'}$. The sign of $d_{AA'}$ indicates which of the two sheets must be taken.

Repeating the same procedure for two other pairs of sites (B, B') and (C, C'), we get in all three hyperboloids on which the object X must lie and its position can be therefore determined as their intersection. The space coordinates $[x_1, x_2, x_3]$ of X are then computed from the measured distance differences $d_{AA'}$, $d_{BB'}$ and $d_{CC'}$.

The difference vector $[d_{AA'}, d_{BB'}, d_{CC'}]$ can be easily computed from $[x_1, x_2, x_3]$, and the corresponding mapping $F : [x_1, x_2, x_3] \to [d_{AA'}, d_{BB'}, d_{CC'}]$ can be explicitly expressed. If the sites A, A' have the space coordinates $[a_1, a_2, a_3]$ and $[a'_1, a'_2, a'_3]$ respectively, then for example

$$\begin{aligned} d_{AA'} = & \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} \\ & -\sqrt{(x_1 - a'_1)^2 + (x_2 - a'_2)^2 + (x_3 - a'_3)^2}. \end{aligned}$$

On the other hand the inversion mapping F^{-1} can not be in general expressed explicitly and the position of X must be computed from $[d_{AA'}, d_{BB'}, d_{CC'}]$ numerically as a solution of a system of algebraic equations of the total degree 8.

In practice a network of observation sites should be used. But the smallest operational system consists of four sites only. In this case one site $O = A' = B' = C'$ is considered as central one and the position of the object X is computed from the distance differences $[d_{AO}, d_{BO}, d_{CO}]$. In the sequel we will restrict ourselves to this simplest case. As we will show, in this case an explicit inversion formula for F^{-1} can be always given.

1.3 Measurement Error of the POS

Suppose, that a pulse is received at four observation sites O, A, B and C at times t_O, t_A, t_B and t_C. The error of the vector $[t_O, t_A, t_B, t_C]$ of independently measured times can be well modeled by a multivariate normal distribution, characterized by its mean value $[0, 0, 0, 0]$ and the variation-covariation matrix having on the diagonal the variations of the time errors at the four sites, which are not necessarily the same

$$\begin{bmatrix} {\sigma_O}^2 & 0 & 0 & 0 \\ 0 & {\sigma_A}^2 & 0 & 0 \\ 0 & 0 & {\sigma_B}^2 & 0 \\ 0 & 0 & 0 & {\sigma_C}^2 \end{bmatrix}. \tag{1}$$

The differences d_{AO}, d_{BO} and d_{CO} have no more independent errors, but the error of the vector $[d_{AO}, d_{BO}, d_{CO}]$ has still a normal distribution characterized by its mean value $[0, 0, 0]$ and the variation-covariation matrix

$$c^2 \begin{bmatrix} {\sigma_A}^2 + {\sigma_O}^2 & {\sigma_O}^2 & {\sigma_O}^2 \\ {\sigma_O}^2 & {\sigma_B}^2 + {\sigma_O}^2 & {\sigma_O}^2 \\ {\sigma_O}^2 & {\sigma_O}^2 & {\sigma_C}^2 + {\sigma_O}^2 \end{bmatrix}, \tag{2}$$

where c is the speed of light. See [3] for the details about multivariate distributions and their characteristics.

If we compute the position $[x_1, x_2, x_3]$ using d_{AO}, d_{BO} and d_{CO} we transform the error distribution by the mapping F^{-1}. The transformed distribution will be no more normal. For this reason the mean value and the variation-covariation matrix are no more sufficient characteristics of this transformed error distribution.

In fact the analysis of such complex multivariate distributions is a difficult problem. This is due to the fact that the standard concepts used in in the case of one dimensional distributions, are insufficient for the description of the geometry of the multivariate distributions. We are convinced that the methods of the applied geometry would be very useful in the analysis of both theoretical distributions and experimental data. See [2] for one possible approach based on the concept of the data depth.

1.4 Confidence Sets

The confidence sets (called also tolerance regions) are perhaps geometrically the most interesting characteristics of probability distributions.

Definition 1. *For a given random variable U having the density function p_U and for a given probability $\alpha \in (0, 1]$ we define the confidence set $C_{U,\alpha}$ as a region for which*

$$\int_{x \in C_{U,\alpha}} p_U(x) = \alpha \tag{3}$$

In other words a confidence set is a region in which the random variable U lies with the probability α. In practice α is set quite high, for example 0.99, and thus a confidence set is simply a region in which the random variable lies with a reasonable certitude.

It is clear from the definition, that for a given probability $\alpha < 1$ there is in general more then one confidence set. There are natural additional properties which can be required of the confidence sets. First of all the confidence sets should be as small as possible in order to give good information about the probability density. For the same reason their boundaries should be the iso-lines (iso-surfaces) of the density function. In the case of multivariate normal distributions it is customary to use suitable ellipsoids as confidence sets. These ellipsoids satisfy both additional requirements (see for example [3, 45.9]).

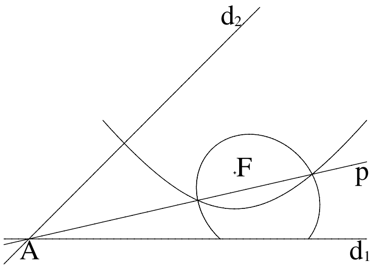
The distribution of the error of the vector $[d_{AO}, d_{BO}, d_{CO}]$ can be described by a system of ellipsoids (confidence sets) depending on the probability α and on the values $[d_{AO}, d_{BO}, d_{CO}]$ (the error may in general depend on the value of $[d_{AO}, d_{BO}, d_{CO}]$). Transforming this system by F^{-1} we will get a new system of confidence sets describing the error of the position $[x_1, x_2, x_3]$. The boundaries of these new confidence sets will be iso-surfaces of the new density function.

2 Explicit Inversion Formula

The importance of an explicit formula for F^{-1} is obvious from the previous section. As X is obtained as intersection of three quadric surfaces, the resulting system of equations has degree 8. Therefore there is seemingly no possibility to obtain an explicit expression for F^{-1}. However for concrete examples, we were able to reduce the degree of the problem and even obtain a simple explicit formula. A deeper investigation of this fact has shown, that this simplification is due to the following interesting property.

2.1 Intersection of Quadric Surfaces of Revolution

Proposition 2. *Let S_1, S_2 be two quadric surfaces of revolution, each of which obtained by rotating a conic section around its main axis. (The only axis for a parabola and the axis passing through the foci for an ellipse or an hyperbola.) Suppose that S_1 and S_2 have a common focus. Then their intersection can be decomposed into curves of degree 2.*



Proof. Let F be the common focus. Clearly the axes of S_1 and S_2 intersect in the point F and therefore they lie in a plane. The previous fig. represents this plane and its intersections with all mentioned objects.

We can characterize the surfaces S_1 and S_2 using the focus-directrix property of the generating conic sections. Obviously the surface S_1 is precisely the set of points in the space, having a constant ratio of distances to the focus F and a directrix plane d_1, perpendicular to the main axis: $S_1 = \{X, \frac{|XF|}{|Xd_1|} = r_1\}$ for some fixed ratio r_1. For $r_1 = 1$ we get a paraboloid, for $r_1 < 1$ an ellipsoid and for $r_1 > 1$ a two-sheet hyperboloid. In the same way the surface S_2 can be characterized as the set $S_2 = \{X, \frac{|XF|}{|Xd_2|} = r_2\}$ for some plane d_2 perpendicular to the axis of S_2 and for some fixed ratio r_2.

For the points of the intersection $X \in S_1 \cap S_2$ we thus get $\frac{|Xd_1|}{|Xd_2|} = \frac{r_2}{r_1}$. This equality characterizes all the points lying in two planes passing through the intersection $d_1 \cap d_2$. One of these planes is denoted p on the figure. As the intersection of a quadric surface with a plane is of degree 2, the intersection $S_1 \cap S_2$ must have a component of degree 2. As $S_1 \cap S_2$ is itself of degree 4, the proposition is proved. □

So the intersection of two hyperboloids, which is in general a curve of degree 4, will have components of degree 2 (conic sections) if the two hyperboloids share a focus.

Consequently the degree 8 system describing a general POS will be decomposed if two of hyperboloids have a common focus. If particular if the three hyperboloids have a common focus - the site O - the problem will be reduced twice and the resulting system can be decomposed to the degree 2 systems. In this case therefore an explicit inversion formula can be always obtained.

We will not describe this explicit formula in general, but we will study in detail one particular example. In this example the situation is simplified even more by the additional condition, that the four sites A, B, C, O are coplanar.

2.2 Example

Let us consider the POS in which the four sites lie in one plane and have the coordinates: $O = [0,0,0]$, $A = [30,0,0]$, $B = [-15,26,0]$ and $C = [-15,-26,0]$. The mapping F is then expressed by formulae:

$$\begin{aligned} d_{AO} &= \sqrt{x_1^2 - 60x_1 + 900 + x_2^2 + x_3^2} - \sqrt{x_1^2 + x_2^2 + x_3^2} \\ d_{BO} &= \sqrt{x_1^2 + 30x_1 + 901 + x_2^2 - 52x_2 + x_3^2} - \sqrt{x_1^2 + x_2^2 + x_3^2} \\ d_{CO} &= \sqrt{x_1^2 + 30x_1 + 901 + x_2^2 + 52x_2 + x_3^2} - \sqrt{x_1^2 + x_2^2 + x_3^2} \end{aligned} \tag{4}$$

We implicitise these equations and obtain implicit algebraic equations of the three hyperboloids H_{AO}, H_{BO} and H_{CO}. For example the implicit equation of H_{AO} is

$$4d_{AO}^2\left(x_1^2 + x_2^2 + x_3^2\right) - \left(900 - 60x_1 - d_{AO}^2\right)^2 = 0.$$

Due to the Proposition 2 any two of these hyperboloids intersect in two conic sections. Because of the symmetry with regard to the plane $x_3 = 0$ these conics lie in the planes perpendicular to the plane $x_3 = 0$. Their projections to this plane will be therefore lines.

For the determination of the 8 intersections of the hyperboloids H_{AO}, H_{BO} and H_{CO} we first evaluate the resultant with respect to x_3 of the implicit equations of H_{AO} and H_{BO}. Because of the previous observations this resultant (of degree 4 in x_1, x_2) can be factorised in two linear factors (each of them of with multiplicity two) describing two stright lines p_1 and p_2. In a similar way from the equations of H_{AO} and H_{CO} we get two lines q_1 and q_2. As intersection of this two pairs of lines we get four points $X_{i,j} = p_i \cap q_j$, $i, j = 1..2$, each of them being projection of two symetrical intersections of the three hyperboloids. The signs of d_{AO}, d_{AO} and d_{AO} will indicate which of the four points $X_{i,j}$ must be taken. The last coordinate x_3^i can be calculated from the equation of any of the three hyperboloids.

Let us give the explicit formula of one of the 4 pairs of solutions of our example system (4):

$$x_1 = \frac{d_{AO}(d_{BO}^2 + d_{CO}^2 - 1802) + (900 - d_{AO}^2)(d_{BO} + d_{CO})}{60(d_{AO} + d_{BO} + d_{CO})} \tag{5}$$

$$x_2 = \frac{d_{AO}(d_{CO}^2 - d_{BO}^2) + (d_{AO}^2 - 2d_{BO}d_{CO} - 2702)(d_{BO} - d_{CO})}{104(d_{AO} + d_{BO} + d_{CO})} \tag{6}$$

$$x_3 = \pm\frac{\sqrt{P_6(d_{AO}, d_{BO}, d_{CO})}}{d_{AO} + d_{BO} + d_{CO}} \tag{7}$$

where $P_6(d_{AO}, d_{BO}, d_{CO})$ is a polynomial of degree 6 in d_{AO}, d_{BO} and d_{CO}. The x_3 is usually supposed to be positiv, as the object (plane) is usually "over" the observation sites.

A similar explicit form of F^{-1} can be in general obtained for any POS having the four sites in a plane. In this case, the first two coordites x_1, x_2 can be expressed as rational functions in d_{AO}, d_{BO} and d_{CO}, but the expression of x_3 will involve a square root.

If the four sites are not coplanar, an explicit formula can be still obtained, but square roots will appear in the expressions of all coordinates.

For a general POS, based on three independent pairs of sites (A, A'), (B, B') and (C, C'), no closed expression of F^{-1} can be obtained.

3 Approximate Representation

The explicit inversion formula is not available for the POS in the general position. In some other cases the inversion formula can be too complicated. For this reason we will describe in this section a general method for the approximation of F^{-1}.

3.1 General Setting

Let us consider the following general setting. Suppose that $x = [x_1, \ldots, x_n]$ is a set of parameters which is transformed by a local diffeomorphism F to a second set of parameters $y = [y_1, \ldots, y_n]$:

$$F : [x_1, \ldots, x_n] \to [y_1, \ldots, y_n] \tag{8}$$

Suppose in addition that an algebraic implicitisation of F is available. We mean by this a system of algebraic equations

$$G(x, y) = 0 \tag{9}$$

which hold if and only if $y = F(x)$.

Next suppose that in the space of parameters y the system of confidence sets (for example a system of ellipsoids) is described. We want to obtain a description of the transformed system of the confidence sets in the space of the parameters x.

3.2 Implicit Representation

If the confidence sets in the space of parameters y are described implicitly we can obtain an implicit description in the space of parameters x in a straightforward way. Suppose, that the boundaries of the confidence sets in the space of parameters y are given by implicit equations

$$E_{\alpha,\overline{y}}(y) = 0 \tag{10}$$

depending algebraically on the measured value $\overline{y}$. Then substituting $y = F(x)$ and $\overline{y} = F(\overline{x})$ in this equations we get mplicit representations of the boundaries of the confidence sets in the space x depending on $\overline{x}$.

The drawbacks of this methods are obvious. As the transformation F is not necessarily rational, we obtain in general a complicated (non algebraic) implicit representation depending in a complicated way on $\overline{x}$.

3.3 Approximation by the Taylor Expansion

Another natural possibility is to approximate the inversion F^{-1} by its Taylor expansion in a suitable point $\overline{y}$:

$$F^{-1}(y) = F^{-1}(\overline{y}) + D_1 F_{\overline{y}}^{-1}(y-\overline{y}) + \frac{1}{2} D_2 F_{\overline{y}}^{-1}(y-\overline{y}) + \frac{1}{6} D_3 F_{\overline{y}}^{-1}(y-\overline{y}) + \dots \quad (11)$$

where $D_i F_{\overline{y}}^{-1}$ is the i-th total differential of F^{-1} at the point $\overline{y}$. See [4, par. 3.14] for the details about the multivariate Taylor expansion. The value of $\overline{x} = F^{-1}(\overline{y})$ can be calculated numerically from (9) and the operators $D_i F_{\overline{y}}^{-1}$ can be obtained by the implicit differentiation of (9), or from the known partial derivatives of F at the point $\overline{x}$. This approximation can be used for an approximate representation of the confidence sets in the space of parameters x. In particular if we have a parametrisation of the boundaries of the confidence sets in the space of parameters y, we can compose this parametrisation with the Taylor expansion and this way obtain an approximate parametrisation of the boundaries of the confidence sets in the space of parameters x.

The disadvantage of this approach is that the Taylor expansion can give a sufficiently good approximation in the proximity of the point $\overline{y}$ but will not be sufficient for more distant points.

3.4 Symbolic Computation of the Taylor Expansion

We propose a different approach, which consists in the symbolic computation of the Taylor expansion simultaneously in all points. If the mapping F^{-1} can not be expressed explicitly, there is no hope to get a general expression of the Taylor expansion depending on the point $\overline{y}$. On the other hand it is possible to get such general expression depending on the target point $\overline{x} = F^{-1}(\overline{y})$.

The total differentials $D_i F_{\overline{y}}^{-1}$ can be symbolically computed via partial differentiation of the equality

$$G(F^{-1}(y), y) = 0 \quad (12)$$

For example by taking all the partial derivatives of the first order $\frac{\partial}{dy_i}$ for $i = 1..n$, we obtain a system of n linear equations for n unknown partial derivatives $\frac{\partial F^{-1}}{dy_i}$. The coefficients of these equations are polynomials in y and $F^{-1}(y)$. This system can be symbolically solved and we get $\frac{\partial F^{-1}}{dy_i}$ in the form of a rational function of y and $F^{-1}(y)$. If we use in a similar way the higher partial derivatives of (12), we get the same king of expression for the higher partial derivatives. See [4, par. 4.5] for more details about the implicit differentiation.

Substituting these expressions into (11), we obtain the Taylor expansion having all the coefficients dependent rationally on $\overline{y}$ and $F^{-1}(\overline{y})$. In this expression we can simply substitute $F(\overline{x})$ for $\overline{y}$ and $\overline{x}$ for $F^{-1}(\overline{y})$ and we obtain the desired simultaneous Taylor expansion depending on $\overline{x}$.