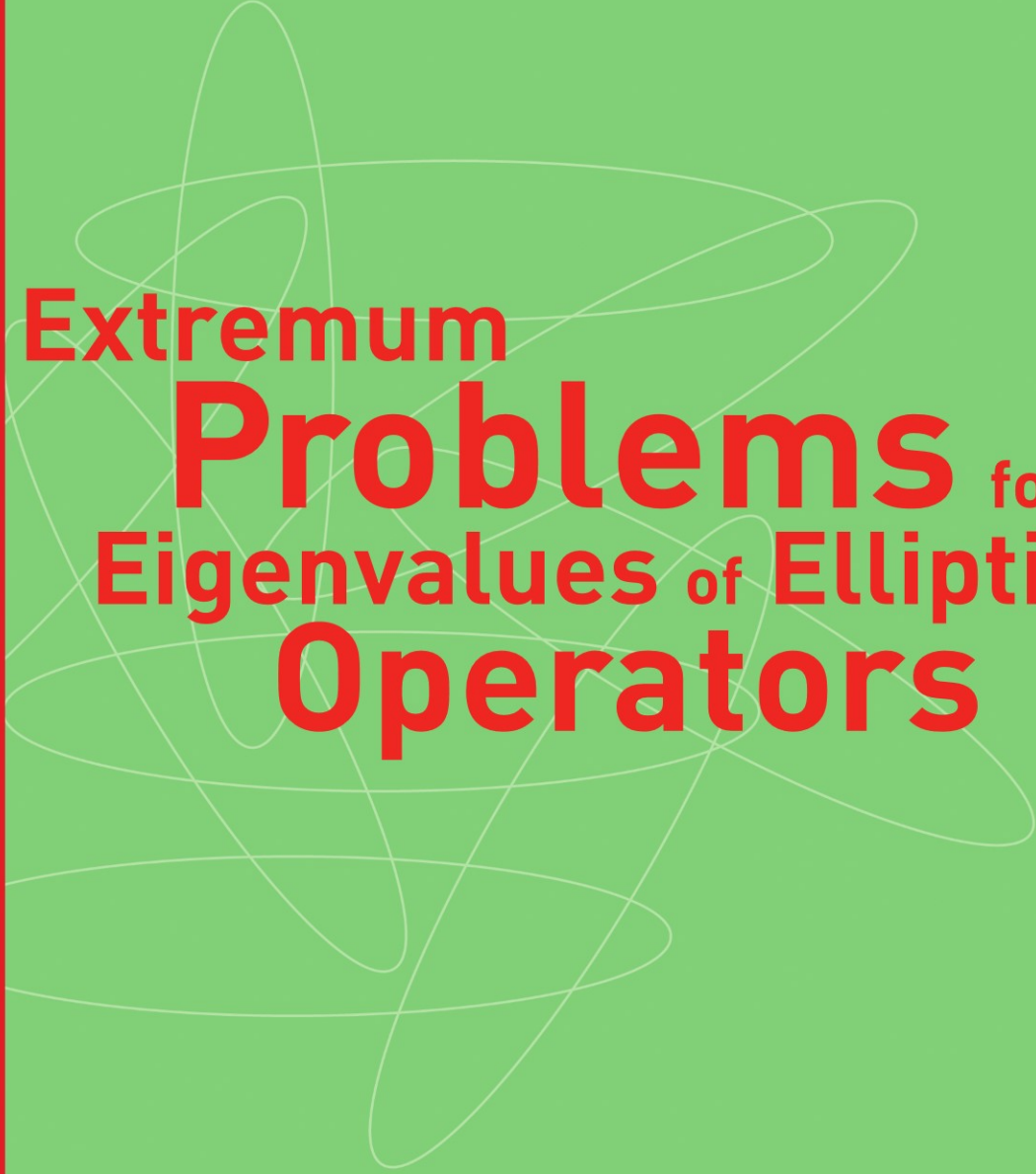


Antoine Henrot

Extremum

Problems for
Eigenvalues of Elliptic
Operators

Antoine Henrot



Extremum
Problems for
Eigenvalues of Elliptic
Operators

Frontiers in Mathematics

Birkhäuser

Antoine Henrot

Extremum
Problems for
Eigenvalues of Elliptic
Operators

Birkhäuser Verlag
Basel · Boston · Berlin

Author's address:

Antoine Henrot
Institut Elie Cartan Nancy
Nancy-Université – CNRS – INRIA
BP 239 54506 Vandoeuvre-les-Nancy Cedex
France
e-mail: henrot@iecn.u-nancy.fr

2000 Mathematical Subject Classification 35P05, 35P15, 35B37, 35J05, 35J10,
49J20, 49K20, 49Q10

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

ISBN 3-7643-7705-4 Birkhäuser Verlag, Basel – Boston – Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part
of the material is concerned, specifically the rights of translation, reprinting, re-use
of illustrations, recitation, broadcasting, reproduction on microfilms or in other ways,
and storage in data banks. For any kind of use permission of the copyright owner
must be obtained.

© 2006 Birkhäuser Verlag, P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media

Cover design: Birgit Blohmann, Zürich, Switzerland

Printed on acid-free paper produced from chlorine-free pulp. TCF ∞

Printed in Germany

ISBN 10: 3-7643-7705-4

e-ISBN: 3-7643-7706-2

ISBN 13: 978-3-7643-7705-2

9 8 7 6 5 4 3 2 1

www.birkhauser.ch

Contents

Preface	ix
1 Eigenvalues of elliptic operators	1
1.1 Notation and prerequisites	1
1.1.1 Notation and Sobolev spaces	1
1.1.2 Partial differential equations	2
1.2 Eigenvalues and eigenfunctions	4
1.2.1 Abstract spectral theory	4
1.2.2 Application to elliptic operators	5
1.2.3 First Properties of eigenvalues	8
1.2.4 Regularity of eigenfunctions	9
1.2.5 Some examples	9
1.2.6 Fredholm alternative	11
1.3 Min-max principles and applications	12
1.3.1 Min-max principles	12
1.3.2 Monotonicity	13
1.3.3 Nodal domains	14
1.4 Perforated domains	15
2 Tools	17
2.1 Schwarz rearrangement	17
2.2 Steiner symmetrization	18
2.2.1 Definition	18
2.2.2 Properties	20
2.2.3 Continuous Steiner symmetrization	21
2.3 Continuity of eigenvalues	23
2.3.1 Introduction	23
2.3.2 Continuity with variable coefficients	26
2.3.3 Continuity with variable domains (Dirichlet case)	28
2.3.4 The case of Neumann eigenvalues	33
2.4 Two general existence theorems	35
2.5 Derivatives of eigenvalues	37

2.5.1	Introduction	37
2.5.2	Derivative with respect to the domain	38
2.5.3	Case of multiple eigenvalues	41
2.5.4	Derivative with respect to coefficients	43
3	The first eigenvalue of the Laplacian-Dirichlet	45
3.1	Introduction	45
3.2	The Faber-Krahn inequality	45
3.3	The case of polygons	46
3.3.1	An existence result	47
3.3.2	The cases $N = 3, 4$	50
3.3.3	A challenging open problem	51
3.4	Domains in a box	52
3.5	Multi-connected domains	55
4	The second eigenvalue of the Laplacian-Dirichlet	61
4.1	Minimizing λ_2	61
4.1.1	The Theorem of Krahn-Szegö	61
4.1.2	Case of a connectedness constraint	63
4.2	A convexity constraint	63
4.2.1	Optimality conditions	64
4.2.2	Geometric properties of the optimal domain	67
4.2.3	Another regularity result	71
5	The other Dirichlet eigenvalues	73
5.1	Introduction	73
5.2	Connectedness of minimizers	74
5.3	Existence of a minimizer for λ_3	76
5.3.1	A concentration-compactness result	76
5.3.2	Existence of a minimizer	77
5.4	Case of higher eigenvalues	80
6	Functions of Dirichlet eigenvalues	85
6.1	Introduction	85
6.2	Ratio of eigenvalues	86
6.2.1	The Ashbaugh-Benguria Theorem	86
6.2.2	Some other ratios	90
6.2.3	A collection of open problems	92
6.3	Sums of eigenvalues	93
6.3.1	Sums of eigenvalues	93
6.3.2	Sums of inverses	94
6.4	General functions of λ_1 and λ_2	95
6.4.1	Description of the set $\mathcal{E} = (\lambda_1, \lambda_2)$	95
6.4.2	Existence of minimizers	98

7	Other boundary conditions for the Laplacian	101
7.1	Neumann boundary condition	101
7.1.1	Introduction	101
7.1.2	Maximization of the second Neumann eigenvalue	102
7.1.3	Some other problems	104
7.2	Robin boundary condition	106
7.2.1	Introduction	106
7.2.2	The Bossel-Daners Theorem	107
7.2.3	Optimal insulation of conductors	110
7.3	Stekloff eigenvalue problem	113
8	Eigenvalues of Schrödinger operators	117
8.1	Introduction	117
8.1.1	Notation	117
8.1.2	A general existence result	119
8.2	Maximization or minimization of the first eigenvalue	119
8.2.1	Introduction	119
8.2.2	The maximization problem	119
8.2.3	The minimization problem	123
8.3	Maximization or minimization of other eigenvalues	125
8.4	Maximization or minimization of the fundamental gap $\lambda_2 - \lambda_1$	127
8.4.1	Introduction	127
8.4.2	Single-well potentials	127
8.4.3	Minimization or maximization with an L^∞ constraint	131
8.4.4	Minimization or maximization with an L^p constraint	134
8.5	Maximization of ratios	136
8.5.1	Introduction	136
8.5.2	Maximization of $\lambda_2(V)/\lambda_1(V)$ in one dimension	136
8.5.3	Maximization of $\lambda_n(V)/\lambda_1(V)$ in one dimension	137
9	Non-homogeneous strings and membranes	141
9.1	Introduction	141
9.2	Existence results	143
9.2.1	A first general existence result	143
9.2.2	A more precise existence result	143
9.2.3	Nonlinear constraint	146
9.3	Minimizing or maximizing $\lambda_k(\rho)$ in dimension 1	148
9.3.1	Minimizing $\lambda_k(\rho)$	149
9.3.2	Maximizing $\lambda_k(\rho)$	150
9.4	Minimizing or maximizing $\lambda_k(\rho)$ in higher dimension	152
9.4.1	Case of a ball	152
9.4.2	General case	153
9.4.3	Some extensions	155

10 Optimal conductivity	159
10.1 Introduction	159
10.2 The one-dimensional case	160
10.2.1 A general existence result	160
10.2.2 Minimization or maximization of $\lambda_k(\sigma)$	161
10.2.3 Case of Neumann boundary conditions	163
10.3 The general case	165
10.3.1 The maximization problem	165
10.3.2 The minimization problem	168
11 The bi-Laplacian operator	169
11.1 Introduction	169
11.2 The clamped plate	169
11.2.1 History	169
11.2.2 Notation and statement of the theorem	170
11.2.3 Proof of the Rayleigh conjecture in dimension $N = 2, 3$	171
11.3 Buckling of a plate	174
11.3.1 Introduction	174
11.3.2 The case of a positive eigenfunction	175
11.3.3 An existence result	177
11.3.4 The last step in the proof	178
11.4 Some other problems	181
11.4.1 Non-homogeneous rod and plate	181
11.4.2 The optimal shape of a column	183
References	187
Index	199

Preface

Problems linking the shape of a domain or the coefficients of an elliptic operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that they involve different fields of mathematics: spectral theory, partial differential equations, geometry, calculus of variations Moreover, they are very simple to state and generally hard to solve! In particular, one can find in the next pages more than 30 open problems!

In this book, we focus on extremal problems. For instance, we look for a domain which minimizes or maximizes a given eigenvalue of the Laplace operator with various boundary conditions and various geometric constraints. We also consider the case of functions of eigenvalues. We investigate similar questions for other elliptic operators, like Schrödinger, non-homogeneous membranes or composites.

The targeted audience is mainly pure and applied mathematicians, more particularly interested in partial differential equations, calculus of variations, differential geometry, spectral theory. More generally, people interested in properties of eigenvalues in other fields such as acoustics, theoretical physics, quantum mechanics, solid mechanics, could find here some answers to natural questions. For that purpose, I choose to recall basic facts and tools in the two first chapters (with only a few proofs). In chapters 3, 4 and 5, we present known results and open questions for the minimization problem of a given eigenvalue $\lambda_k(\Omega)$ of the Laplace operator with Dirichlet boundary conditions, where the unknown is here the domain Ω itself. In chapter 6, we investigate various functions of the Dirichlet eigenvalues, while chapter 7 is devoted to eigenvalues of the Laplace operator with other boundary conditions. In chapter 8, we consider the eigenvalues of Schrödinger operators: therefore, the unknown is no longer the shape of the domain but the potential V . Chapter 9 is devoted to non-homogeneous membranes and chapter 10 to more general elliptic operators in divergence form. At last, in chapter 11, we are interested in the bi-Laplace operator.

Of course no book can completely cover such a huge field of research. In making personal choices for inclusion of material, I tried to give useful complementary references, in the process certainly neglecting some relevant works. I would be grateful to hear from readers about important missing citations.

I would like to thank Benoit Perthame who suggested in September 2004 that I write this book. Many people helped me with the enterprise, answering my questions and queries or suggesting interesting problems: Mark Ashbaugh, Friedemann Brock, Dorin Bucur, Giuseppe Buttazzo, Steve Cox, Pedro Freitas, Antonio Greco, Evans Harrell, Francois Murat, Edouard Oudet, Gerard Philippin, Michel Pierre, Marius Tucsnak. I am pleased to thank them here.

Nancy, March 2006

Antoine Henrot

Chapter 1

Eigenvalues of elliptic operators

1.1 Notation and prerequisites

In this section, we recall the basic results of the theory of elliptic partial differential equations. The prototype of elliptic operator is the Laplacian, but the results that we state here are also valid for more general (linear) elliptic operators. For the basic facts we recall here, we refer to any textbook on partial differential equations and operator theory. For example, [36], [58], [75], [83] are good standard references.

1.1.1 Notation and Sobolev spaces

Let Ω be a bounded open set in \mathbb{R}^N . We denote by $L^2(\Omega)$ the Hilbert space of square summable functions defined on Ω and by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives (in the sense of distributions) are in $L^2(\Omega)$:

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, \dots, N \right\}.$$

This is a Hilbert space when it is endowed with the scalar product

$$(u, v)_{H^1} := \int_{\Omega} u(x)v(x) dx + \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx$$

and the corresponding norm:

$$\|u\|_{H^1} := \left(\int_{\Omega} u(x)^2 dx + \int_{\Omega} |\nabla u(x)|^2 dx \right)^{1/2}.$$

In the case of Dirichlet boundary conditions, we will use the subspace $H_0^1(\Omega)$ which is defined as the closure of C^∞ functions compactly supported in Ω (functions in $C_0^\infty(\Omega)$) for the norm $\|\cdot\|_{H^1}$. It is also a Hilbert space. At last, $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. For some non-linear problems, for example when we are interested in the p -Laplace operator, it is more convenient to work with the spaces L^p , $p \geq 1$ instead of L^2 . In this case, the Sobolev spaces, defined exactly in the same way, are denoted by $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. These are Banach spaces.

When Ω is bounded (or bounded in one direction), we have the Poincaré inequality:

$$\exists C = C(\Omega) \text{ such that } \forall u \in H_0^1(\Omega), \int_{\Omega} u(x)^2 dx \leq C \int_{\Omega} |\nabla u(x)|^2 dx. \quad (1.1)$$

Actually the constant C which appears in (1.1) is closely related to the eigenvalues of the Laplacian since we will see later (cf (1.36)) that the best possible constant C is nothing other than $1/\lambda_1(\Omega)$ where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

By definition, $H_0^1(\Omega)$ and $H^1(\Omega)$ are continuously embedded in $L^2(\Omega)$, but we will need later a compact embedding. This is the purpose of the following theorem.

Theorem 1.1.1 (Rellich).

- For any bounded open set Ω , the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.
- If Ω is a bounded open set with Lipschitz boundary, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Remark 1.1.2. We can weaken the assumption of Lipschitz boundary but not too much, see e.g. the book [148] for more details.

1.1.2 Partial differential equations

Elliptic operator

Let $a_{ij}(x)$, $i, j = 1, \dots, N$ be bounded functions defined on Ω and satisfying the usual ellipticity assumption:

$$\exists \alpha > 0, \text{ such that } \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \forall x \in \Omega \quad (1.2)$$

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2$$

where $|\xi| = (\xi_1^2 + \xi_2^2 + \dots + \xi_N^2)^{1/2}$ denotes the euclidean norm of the vector ξ . We will also assume a symmetry assumption for the a_{ij} namely:

$$\forall x \in \Omega, \forall i, j \quad a_{ij}(x) = a_{ji}(x). \quad (1.3)$$

Let $a_0(x)$ be a bounded function defined on Ω . We introduce the linear elliptic operator L , defined on $H^1(\Omega)$ by:

$$Lu := - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(x)u \quad (1.4)$$

(derivatives are to be understood in the sense of distributions). The prototype of elliptic operator is the Laplacian:

$$-\Delta u := - \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} \quad (1.5)$$

which will be considered in the main part of this book (chapters 3 to 7). In chapter 8, we consider the Schrödinger operator $L_V u = -\Delta u + V(x)u$ where V (the potential) is a bounded function, while chapters 9 and 10 deal with more general elliptic operators. In that case, we will keep the notation L when we want to consider general operators given by (1.4). At last, in chapter 11, we consider operators of fourth order.

Remark 1.1.3. Let us remark that, since we are only interested in eigenvalue problems, we do not put any sign condition on the function $a_0(x)$ which appears in (1.4). Indeed, since $a_0(x)$ is bounded, we can always replace the operator L by $L + (\|a_0\|_\infty + 1)Id$, i.e. replace the function $a_0(x)$ by $a_0(x) + \|a_0\|_\infty + 1$ if we need a positive function in the term of order 0 of the operator L . For the eigenvalues, that would just induce a translation of $\|a_0\|_\infty + 1$ to the right.

Dirichlet boundary condition

Let f be a function in $L^2(\Omega)$. When we call u a solution of the Dirichlet problem

$$\begin{aligned} Lu &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.6)$$

we actually mean that u is the unique solution of the variational problem

$$\left\{ \begin{array}{l} u \in H_0^1(\Omega) \text{ and } \forall v \in H_0^1(\Omega), \\ \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x)u(x)v(x) dx = \int_{\Omega} f(x)v(x) dx. \end{array} \right. \quad (1.7)$$

Existence and uniqueness of a solution for problem (1.7) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the Poincaré inequality (1.1). Note that, according to Remark 1.1.3, we can restrict ourselves to the case $a_0(x) \geq 0$. In the sequel, we will denote by A_L^D (or $A_L^D(\Omega)$ when we want to emphasize the dependence on the domain Ω) the linear operator defined by:

$$\begin{aligned} A_L^D : L^2(\Omega) &\rightarrow H_0^1(\Omega) \subset L^2(\Omega), \\ f &\mapsto u \text{ solution of (1.7)}. \end{aligned} \quad (1.8)$$

Neumann boundary condition

In the same way, if f is a function in $L^2(\Omega)$, we will also consider u a solution of the Neumann problem

$$\begin{aligned} Lu &= f && \text{in } \Omega, \\ \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} n_i &= 0 && \text{on } \partial\Omega \end{aligned} \quad (1.9)$$

(where n stands for the exterior unit normal vector to $\partial\Omega$ and n_i is its i th coordinate). For example, when $L = -\Delta$, the boundary condition reads (formally)

$$\frac{\partial u}{\partial n} = 0.$$

It means that u is the unique solution in $H^1(\Omega)$ of the variational problem

$$\left\{ \begin{array}{l} u \in H^1(\Omega) \text{ and } \forall v \in H^1(\Omega), \\ \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x) u(x) v(x) dx = \int_{\Omega} f(x) v(x) dx. \end{array} \right. \quad (1.10)$$

Existence and uniqueness of a solution for problem (1.10) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the fact that we can assume that $a_0(x) \geq 1$ (according to Remark 1.1.3). In the sequel, we will denote by A_L^N the linear operator defined by:

$$\begin{aligned} A_L^N : L^2(\Omega) &\rightarrow H^1(\Omega) \subset L^2(\Omega), \\ f &\mapsto u \text{ solution of (1.10)}. \end{aligned} \quad (1.11)$$

Remark 1.1.4. We will also consider later, for example in chapter 7, other kinds of boundary conditions like Robin or Stekloff boundary conditions.

1.2 Eigenvalues and eigenfunctions**1.2.1 Abstract spectral theory**

Let us now give the abstract theorem which provides the existence of a sequence of eigenvalues and eigenfunctions. Let H be a Hilbert space endowed with a scalar product (\cdot, \cdot) and recall that an operator T is a linear continuous map from H into H . We say that:

- T is positive if, $\forall x \in H$, $(Tx, x) \geq 0$,
- T is self-adjoint, if $\forall x, y \in H$, $(Tx, y) = (x, Ty)$,
- T is compact, if the image of any bounded set is relatively compact (i.e. has a compact closure) in H .

Theorem 1.2.1. *Let H be a separable Hilbert space of infinite dimension and T a self-adjoint, compact and positive operator. Then, there exists a sequence of real positive eigenvalues (ν_n) , $n \geq 1$ converging to 0 and a sequence of eigenvectors (x_n) , $n \geq 1$ defining a Hilbert basis of H such that $\forall n, T x_n = \nu_n x_n$.*

Of course, this theorem can be seen as a generalization to Hilbert spaces of the classical result in finite dimension for symmetric or normal matrices (existence of real eigenvalues and of an orthonormal basis of eigenvectors).

1.2.2 Application to elliptic operators

Dirichlet boundary condition

We apply Theorem 1.2.1 to $H = L^2(\Omega)$ and the operator A_L^D defined in (1.8).

- A_L^D is positive: let $f \in L^2(\Omega)$ and $u = A_L^D f$ be the solution of (1.7). We get

$$(f, A_L^D f) = \int_{\omega} f(x)u(x) dx = \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \int_{\Omega} a_0(x)u^2(x) dx .$$

Now, we recall that $a_0(x)$ can be taken as a positive function and then the ellipticity condition (1.2) yields the desired result. Moreover, we see that $(f, A_L^D f) > 0$ as soon as $f \neq 0$ (strict positivity).

- A_L^D is self-adjoint: let $f, g \in L^2(\Omega)$ and $u = A_L^D f$, $v = A_L^D g$. We have:

$$(f, A_L^D g) = \int_{\omega} f(x)v(x)dx = \sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x)u(x)v(x)dx. \quad (1.12)$$

Now, according to the symmetry assumption (1.3) and the equation (1.7) satisfied by v , the right-hand side in (1.12) is equal to $\int_{\Omega} u(x)g(x) dx = (A_L^D f, g)$.

- A_L^D is compact: it is an immediate consequence of the Rellich Theorem 1.1.1.

As a consequence of Theorem 1.2.1, there exists (u_n) a Hilbert basis of $L^2(\Omega)$ and a sequence $\nu_n \geq 0$, converging to 0, such that $A_L^D u_n = \nu_n u_n$. Actually, the ν_n are positive, since the strict positivity of A_L^D yields $\nu_n \|u_n\|_{L^2} = (u_n, A_L^D u_n) > 0$.

Coming back to (1.7), we see that u_n satisfies, $\forall v \in H_0^1(\Omega)$:

$$\nu_n \left(\sum_{i,j=1}^N \int_{\Omega} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\Omega} a_0(x)u_n(x)v(x) dx \right) = \int_{\Omega} u_n(x)v(x) dx$$

which means

$$L u_n = \frac{1}{\nu_n} u_n .$$

Setting $\lambda_n = \frac{1}{\nu_n}$, we have proved: