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Birkhäuser Verlag
Basel · Boston · Berlin

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2000 Mathematical Subject Classification 35P05, 35P15, 35B37, 35J05, 35J10, 49J20, 49K20, 49Q10

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 3-7643-7705-4 Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Cover design: Birgit Blohmann, Zürich, Switzerland
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany
ISBN 10: 3-7643-7705-4 e-ISBN: 3-7643-7706-2
ISBN 13: 978-3-7643-7705-2

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

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Preface

Problems linking the shape of a domain or the coefficients of an elliptic operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that they involve different fields of mathematics: spectral theory, partial differential equations, geometry, calculus of variations .... Moreover, they are very simple to state and generally hard to solve! In particular, one can find in the next pages more than 30 open problems!

In this book, we focus on extremal problems. For instance, we look for a domain which minimizes or maximizes a given eigenvalue of the Laplace operator with various boundary conditions and various geometric constraints. We also consider the case of functions of eigenvalues. We investigate similar questions for other elliptic operators, like Schrödinger, non-homogeneous membranes or composites.

The targeted audience is mainly pure and applied mathematicians, more particularly interested in partial differential equations, calculus of variations, differential geometry, spectral theory. More generally, people interested in properties of eigenvalues in other fields such as acoustics, theoretical physics, quantum mechanics, solid mechanics, could find here some answers to natural questions. For that purpose, I choose to recall basic facts and tools in the two first chapters (with only a few proofs). In chapters 3, 4 and 5, we present known results and open questions for the minimization problem of a given eigenvalue $\lambda_k(\Omega)$ of the Laplace operator with Dirichlet boundary conditions, where the unknown is here the domain Ω itself. In chapter 6, we investigate various functions of the Dirichlet eigenvalues, while chapter 7 is devoted to eigenvalues of the Laplace operator with other boundary conditions. In chapter 8, we consider the eigenvalues of Schrödinger operators: therefore, the unknown is no longer the shape of the domain but the potential V. Chapter 9 is devoted to non-homogeneous membranes and chapter 10 to more general elliptic operators in divergence form. At last, in chapter 11, we are interested in the bi-Laplace operator.

Of course no book can completely cover such a huge field of research. In making personal choices for inclusion of material, I tried to give useful complementary references, in the process certainly neglecting some relevant works. I would be grateful to hear from readers about important missing citations.

I would like to thank Benoit Perthame who suggested in September 2004 that I write this book. Many people helped me with the enterprise, answering my questions and queries or suggesting interesting problems: Mark Ashbaugh, Friedemann Brock, Dorin Bucur, Giuseppe Buttazzo, Steve Cox, Pedro Freitas, Antonio Greco, Evans Harrell, Francois Murat, Edouard Oudet, Gerard Philippin, Michel Pierre, Marius Tucsnak. I am pleased to thank them here.

Nancy, March 2006 *Antoine Henrot*

Chapter 1

Eigenvalues of elliptic operators

1.1 Notation and prerequisites

In this section, we recall the basic results of the theory of elliptic partial differential equations. The prototype of elliptic operator is the Laplacian, but the results that we state here are also valid for more general (linear) elliptic operators. For the basic facts we recall here, we refer to any textbook on partial differential equations and operator theory. For example, [36], [58], [75], [83] are good standard references.

1.1.1 Notation and Sobolev spaces

Let Ω be a bounded open set in $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the Hilbert space of square summable functions defined on Ω and by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives (in the sense of distributions) are in $L^2(\Omega)$:

$$H^1(\Omega) := \{u \in L^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, \ldots, N\}.$$

This is a Hilbert space when it is endowed with the scalar product

$$(u, v)_{H^1} := \int_\Omega u(x)v(x)\,dx + \int_\Omega \nabla u(x).\nabla v(x)\,dx$$

and the corresponding norm:

$$\|u\|_{H^1} := \left(\int_\Omega u(x)^2\,dx + \int_\Omega |\nabla u(x)|^2\,dx\right)^{1/2}.$$

In the case of Dirichlet boundary conditions, we will use the subspace $H_0^1(\Omega)$ which is defined as the closure of C^∞ functions compactly supported in Ω (functions in $C_0^\infty(\Omega)$) for the norm $\| \|_{H^1}$. It is also a Hilbert space. At last, $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. For some non-linear problems, for example when we are interested in the p-Laplace operator, it is more convenient to work with the spaces $L^p, p \geq 1$ instead of L^2. In this case, the Sobolev spaces, defined exactly in the same way, are denoted by $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. These are Banach spaces.

When Ω is bounded (or bounded in one direction), we have the Poincaré inequality:

$$\exists C = C(\Omega) \text{ such that } \forall u \in H_0^1(\Omega), \int_\Omega u(x)^2\,dx \leq C \int_\Omega |\nabla u(x)|^2\,dx\,. \tag{1.1}$$

Actually the constant C which appears in (1.1) is closely related to the eigenvalues of the Laplacian since we will see later (cf (1.36)) that the best possible constant C is nothing other than $1/\lambda_1(\Omega)$ where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

By definition, $H_0^1(\Omega)$ and $H^1(\Omega)$ are continuously embedded in $L^2(\Omega)$, but we will need later a compact embedding. This is the purpose of the following theorem.

Theorem 1.1.1 (Rellich).

- *For any bounded open set Ω, the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*
- *If Ω is a bounded open set with Lipschitz boundary, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Remark 1.1.2. We can weaken the assumption of Lipschitz boundary but not too much, see e.g. the book [148] for more details.

1.1.2 Partial differential equations

Elliptic operator

Let $a_{ij}(x)$, $i, j = 1, \ldots, N$ be bounded functions defined on Ω and satisfying the usual ellipticity assumption:

$$\begin{array}{c}\exists \alpha > 0, \text{ such that } \forall \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N,\ \forall x \in \Omega \\ \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2\end{array} \tag{1.2}$$

where $|\xi| = \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_N^2\right)^{1/2}$ denotes the euclidean norm of the vector ξ. We will also assume a symmetry assumption for the a_{ij} namely:

$$\forall x \in \Omega, \forall i, j \quad a_{ij}(x) = a_{ji}(x)\,. \tag{1.3}$$

Let $a_0(x)$ be a bounded function defined on Ω. We introduce the linear elliptic operator L, defined on $H^1(\Omega)$ by:

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u \tag{1.4}$$

(derivatives are to be understood in the sense of distributions). The prototype of elliptic operator is the Laplacian:

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \tag{1.5}$$

which will be considered in the main part of this book (chapters 3 to 7). In chapter 8, we consider the Schrödinger operator $L_V u = -\Delta u + V(x)u$ where V (the potential) is a bounded function, while chapters 9 and 10 deal with more general elliptic operators. In that case, we will keep the notation L when we want to consider general operators given by (1.4). At last, in chapter 11, we consider operators of fourth order.

Remark 1.1.3. Let us remark that, since we are only interested in eigenvalue problems, we do not put any sign condition on the function $a_0(x)$ which appears in (1.4). Indeed, since $a_0(x)$ is bounded, we can always replace the operator L by $L + (\|a_0\|_\infty + 1)Id$, i.e. replace the function $a_0(x)$ by $a_0(x) + \|a_0\|_\infty + 1$ if we need a positive function in the term of order 0 of the operator L. For the eigenvalues, that would just induce a translation of $\|a_0\|_\infty + 1$ to the right.

Dirichlet boundary condition

Let f be a function in $L^2(\Omega)$. When we call u a solution of the Dirichlet problem

$$\begin{array}{rcll} Lu &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.6}$$

we actually mean that u is the unique solution of the variational problem

$$\left\{ \begin{array}{c} u \in H_0^1(\Omega) \text{ and } \forall v \in H_0^1(\Omega), \\ \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0(x)u(x)v(x)\, dx = \int_\Omega f(x)v(x)\, dx\ . \end{array} \right. \tag{1.7}$$

Existence and uniqueness of a solution for problem (1.7) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the Poincaré inequality (1.1). Note that, according to Remark 1.1.3, we can restrict ourselves to the case $a_0(x) \geq 0$. In the sequel, we will denote by A_L^D (or $A_L^D(\Omega)$ when we want to emphasize the dependence on the domain Ω) the linear operator defined by:

$$\begin{array}{rcl} A_L^D : L^2(\Omega) & \to & H_0^1(\Omega) \subset L^2(\Omega), \\ f & \mapsto & u \text{ solution of (1.7).} \end{array} \tag{1.8}$$

Neumann boundary condition

In the same way, if f is a function in $L^2(\Omega)$, we will also consider u a solution of the Neumann problem

$$\begin{array}{rcl} Lu & = & f \quad \text{in } \Omega, \\ \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} n_i & = & 0 \quad \text{on } \partial\Omega \end{array} \tag{1.9}$$

(where n stands for the exterior unit normal vector to $\partial\Omega$ and n_i is its ith coordinate). For example, when $L = -\Delta$, the boundary condition reads (formally)

$$\frac{\partial u}{\partial n} = 0 \, .$$

It means that u is the unique solution in $H^1(\Omega)$ of the variational problem

$$\begin{cases} u \in H^1(\Omega) \text{ and } \forall v \in H^1(\Omega), \\ \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega a_0(x) u(x) v(x) \, dx = \int_\Omega f(x) v(x) \, dx \, . \end{cases} \tag{1.10}$$

Existence and uniqueness of a solution for problem (1.10) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the fact that we can assume that $a_0(x) \geq 1$ (according to Remark 1.1.3). In the sequel, we will denote by A_L^N the linear operator defined by:

$$\begin{array}{rcl} A_L^N : L^2(\Omega) & \to & H^1(\Omega) \subset L^2(\Omega), \\ f & \mapsto & u \text{ solution of (1.10).} \end{array} \tag{1.11}$$

Remark 1.1.4. We will also consider later, for example in chapter 7, other kinds of boundary conditions like Robin or Stekloff boundary conditions.

1.2 Eigenvalues and eigenfunctions

1.2.1 Abstract spectral theory

Let us now give the abstract theorem which provides the existence of a sequence of eigenvalues and eigenfunctions. Let H be a Hilbert space endowed with a scalar product $(.,.)$ and recall that an operator T is a linear continuous map from H into H. We say that:

- T is positive if, $\forall x \in H, \ (Tx, x) \geq 0$,
- T is self-adjoint, if $\forall x, y \in H, \ (Tx, y) = (x, Ty)$,
- T is compact, if the image of any bounded set is relatively compact (i.e. has a compact closure) in H.

Theorem 1.2.1. *Let H be a separable Hilbert space of infinite dimension and T a self-adjoint, compact and positive operator. Then, there exists a sequence of real positive eigenvalues (ν_n), $n \geq 1$ converging to 0 and a sequence of eigenvectors (x_n), $n \geq 1$ defining a Hilbert basis of H such that $\forall n,\ T\,x_n = \nu_n\,x_n$.*

Of course, this theorem can be seen as a generalization to Hilbert spaces of the classical result in finite dimension for symmetric or normal matrices (existence of real eigenvalues and of an orthonormal basis of eigenvectors).

1.2.2 Application to elliptic operators

Dirichlet boundary condition

We apply Theorem 1.2.1 to $H = L^2(\Omega)$ and the operator A_L^D defined in (1.8).

- A_L^D is positive: let $f \in L^2(\Omega)$ and $u = A_L^D\,f$ be the solution of (1.7). We get

  $$(f, A_L^D\,f) = \int_\omega f(x)u(x)\,dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx + \int_\Omega a_0(x)u^2(x)\,dx\ .$$

  Now, we recall that $a_0(x)$ can be taken as a positive function and then the ellipticity condition (1.2) yields the desired result. Moreover, we see that $(f, A_L^D\,f) > 0$ as soon as $f \neq 0$ (strict positivity).

- A_L^D is self-adjoint: let $f, g \in L^2(\Omega)$ and $u = A_L^D\,f$, $v = A_L^D\,g$. We have:

  $$(f, A_L^D\,g) = \int_\omega f(x)v(x)dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\,dx + \int_\Omega a_0(x)u(x)v(x)dx. \tag{1.12}$$

  Now, according to the symmetry assumption (1.3) and the equation (1.7) satisfied by v, the right-hand side in (1.12) is equal to $\int_\Omega u(x)g(x)\,dx = (A_L^D\,f, g)$.

- A_L^D is compact: it is an immediate consequence of the Rellich Theorem 1.1.1.

As a consequence of Theorem 1.2.1, there exists (u_n) a Hilbert basis of $L^2(\Omega)$ and a sequence $\nu_n \geq 0$, converging to 0, such that $A_L^D\,u_n = \nu_n\,u_n$. Actually, the ν_n are positive, since the strict positivity of A_L^D yields $\nu_n \|u_n\|_{L^2} = (u_n, A_L^D\,u_n) > 0$.

Coming back to (1.7), we see that u_n satisfies, $\forall v \in H_0^1(\Omega)$:

$$\nu_n \left(\sum_{i,j=1}^N \int_\Omega a_{ij}(x)\frac{\partial u_n}{\partial x_i}\,\frac{\partial v}{\partial x_j}\,dx + \int_\Omega a_0(x)u_n(x)v(x)\,dx \right) = \int_\Omega u_n(x)v(x)\,dx$$

which means

$$L\,u_n = \frac{1}{\nu_n}\,u_n\ .$$

Setting $\lambda_n = \frac{1}{\nu_n}$, we have proved: