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Suren Basov

Multi-dimensional Screening

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Studies in Economic Theory

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Multidimensional Screening

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I dedicate this book to my wife Svetlana and my daughter Margaret, with a provision that she will read it when she grows up.

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Preface

In many industries the tariffs are not strictly proportional to the quantity purchased, i. e, they are nonlinear. Examples of nonlinear tariffs include railroad and electricity schedules and rental rates for durable goods and space. The major justification for the nonlinear pricing is the existence of private information on the side of consumers. In the early papers on the subject, private information was captured either by assuming a finite number of types (e. g. Adams and Yellen, 1976) or by a unidimensional continuum of types (Mussa and Rosen, 1978). Economics of the unidimensional problems is by now well understood.

The unidimensional models, however, do not cover all the situations of practical interest. Indeed, often the nonlinear tariffs specify the payment as a function of a variety of characteristics. For example, railroad tariffs specify charges based on weight, volume, and distance of each shipment. Different customers may value each of these characteristics differently, hence the customer's type will not in general be captured by a unidimensional characteristic and a problem of multidimensional screening arises. In such models the consumer's private information (her type) is captured by an $m-$dimensional vector, while the good produced by the monopolist has n quality dimensions.

Recent years witnessed a considerable progress in the area of multidimensional screening. The most important papers are Armstrong (1996), Rochet and Chone (1998), and Basov (2001, 2002, 2004). multidimensional screening models can have fruitful application in the theory of industrial organization, labor markets, and the optimal taxation. For an example of the latter, see Shapiro (2001). However, despite the wide prevalence of non-

linear tariffs and practical importance of understanding the economics that underlies them, this literature remains unknown to the most economists, beyond a circle of a few specialists in the area. In my view, the main obstacle to understanding of this literature is its use of mathematical tools, which are not a part of a standard economist's toolbox. This book attempts to remedy the situation.

I assume that the reader is familiar with basic mathematical tools used by in modern economic theory, such as calculus, linear algebra, elements of the theory of ordinary differential equations, the basics of the measure theory (so I use without a definition such terms as a Borel set, Lebesgue measure, or a Lebesgue measurable function), and the optimal control. I also assume knowledge of microeconomics at the level of Mas-Colell, Whinston, and Green (1995). However, I introduce all more advanced techniques.

The first part of this book contains a review of vector calculus, theory of partial differential equation of the first and second order, and the theory of generalized convexity. These techniques are extensively used in the multidimensional screening models. It also contains a chapter of miscellaneous techniques, which are some times useful in screening models, but are not used as extensively. When the results can be found in easily accessible literature I usually formulate them without a proof, referring the reader to another source. I, however, provide more thorough discussion of the results that are harder to access in the literature.

Part two is devoted to the economics of screening models. It starts with a detailed discussion of economics and mathematics of unidimensional screening problems and three approaches to their solution: direct, dual, and Hamiltonian. It uses the Hamiltonian approach to unify all known results, which were previously obtained using different arguments.

After a thorough discussion of the unidimensional case I move on to the multidimensional screening model. First, I discuss the main economic constraint, which makes the multidimensional case qualitatively different from the unidimensional one: path independence of information rents. I show that it significantly restricts usefulness of the direct approach in the multidimensional case. Dual approach can be taken somewhat further, but it also breaks down, when the dimensionality of type and the number of the screening instruments differ. Hamiltonian approach, however, can be applied fruitfully even in this case. For the most part of the book I, following almost all the literature in the area, assume that the consumer's utility is quasilinear in money. However, in Chapter 8 I go beyond the quasilinear case and provide some results, which apply to a more general specification. I also argue that is some applied models that arise the most natural specification is not quasilinear.

The final chapter concludes and outlines some directions for the future research. One of them, already mentioned, is going beyond the quasilinear case. Another is to look at screening models with finitely many goods and an infinitely dimensional type. I argue that this is probably the most relevant

case from the economic point of view and give some thoughts how the techniques developed in the book can be adapted to handle it.

Each chapter, except for the last one, is followed by exercises. That makes the book useful for instructors, who wish to use it to teach an advanced graduate course in multidimensional mechanism design. Exercises are also important for the general reader, since they allow to develop mathematical and economic intuition and provide deeper understanding of the theoretical facts. Though this book should be sufficient to teach the reader all main mathematical techniques in the area of the multidimensional screening, I provided bibliographic notes in the end of each chapter, so an interested reader can achieve a deeper understanding of the material covered in the chapter and investigate the connections with a broader mechanism design perspective.

I hope that collecting necessary technical tools and economic insights of the multidimensional screening literature in one volume will facilitate the use of the results by applied economists and help to shed some light on many areas of economic theory.

Melbourne
October, 2004

SUREN BASOV

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