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Optimal Control with Engineering Applications

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With 12 Figures

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Library of Congress Control Number: 2007920933

ISBN 978-3-540-69437-3 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springer.com

Typesetting: Camera ready by author
Production: LE-TEX Jelonek, Schmidt & Vöckler GbR, Leipzig
Cover design: eStudioCalamar S.L., F. Steinen-Broo, Girona, Spain

SPIN 11880127 7/3100/YL - 5 4 3 2 1 0 Printed on acid-free paper

Foreword

This book is based on the lecture material for a one-semester senior-year undergraduate or first-year graduate course in optimal control which I have taught at the Swiss Federal Institute of Technology (ETH Zurich) for more than twenty years. The students taking this course are mostly students in mechanical engineering and electrical engineering taking a major in control. But there also are students in computer science and mathematics taking this course for credit.

The only prerequisites for this book are: The reader should be familiar with dynamics in general and with the state space description of dynamic systems in particular. Furthermore, the reader should have a fairly sound understanding of differential calculus.

The text mainly covers the design of open-loop optimal controls with the help of Pontryagin's Minimum Principle, the conversion of optimal open-loop to optimal closed-loop controls, and the direct design of optimal closed-loop optimal controls using the Hamilton-Jacobi-Bellman theory.

In theses areas, the text also covers two special topics which are not usually found in textbooks: the extension of optimal control theory to matrix-valued performance criteria and Lukes' method for the iterative design of approximatively optimal controllers.

Furthermore, an introduction to the phantastic, but incredibly intricate field of differential games is given. The only reason for doing this lies in the fact that the differential games theory has (exactly) one simple application, namely the LQ differential game. It can be solved completely and it has a very attractive connection to the H_∞ method for the design of robust linear time-invariant controllers for linear time-invariant plants. — This route is the easiest entry into H_∞ theory. And I believe that every student majoring in control should become an expert in H_∞ control design, too.

The book contains a rather large variety of optimal control problems. Many of these problems are solved completely and in detail in the body of the text. Additional problems are given as exercises at the end of the chapters. The solutions to all of these exercises are sketched in the Solution section at the end of the book.

Acknowledgements

First, my thanks go to Michael Athans for elucidating me on the background of optimal control in the first semester of my graduate studies at M.I.T. and for allowing me to teach his course in my third year while he was on sabbatical leave.

I am very grateful that Stephan A. R. Hepner pushed me from teaching the geometric version of Pontryagin's Minimum Principle along the lines of [2], [20], and [14] (which almost no student understood because it is so easy, but requires 3D vision) to teaching the variational approach as presented in this text (which almost every student understands because it is so easy and does not require any 3D vision).

I am indebted to Lorenz M. Schumann for his contributions to the material on the Hamilton-Jacobi-Bellman theory and to Roberto Cirillo for explaining Lukes' method to me.

Furthermore, a large number of persons have supported me over the years. I cannot mention all of them here. But certainly, I appreciate the continuous support by Gabriel A. Dondi, Florian Herzog, Simon T. Keel, Christoph M. Schär, Esfandiar Shafai, and Oliver Tanner over many years in all aspects of my course on optimal control. — Last but not least, I like to mention my secretary Brigitte Rohrbach who has always eagle-eyed my texts for errors and silly faults.

Finally, I thank my wife Rosmarie for not killing me or doing any other harm to me during the very intensive phase of turning this manuscript into a printable form.

Hans P. Geering

Fall 2006

Contents

List of Symbols

Independent Variables

t	time
t_a, t_b	initial time, final time
t_1, t_2	times in (t_a, t_b), e.g., starting end ending times of a singular arc
τ	a special time in $[t_a, t_b]$

Vectors and Vector Signals

$u(t)$	control vector, $u(t) \in \Omega \subseteq R^m$
$x(t)$	state vector, $x(t) \in R^n$
$y(t)$	output vector, $y(t) \in R^p$
$y_d(t)$	desired output vector, $y_d(t) \in R^p$
$\lambda(t)$	costate vector, $\lambda(t) \in R^n$, i.e., vector of Lagrange multipliers
q	additive part of $\lambda(t_b) = \nabla_x K(x(t_b)) + q$ which is involved in the transversality condition
λ_a, λ_b	vectors of Lagrange multipliers
$\mu_0, \dots, \mu_{\ell-1}$, $\mu_\ell(t)$	scalar Lagrange multipliers

Sets

$\Omega \subseteq R^m$	control constraint
$\Omega_u \subseteq R^{m_u}$, $\Omega_v \subseteq R^{m_v}$	control constraints in a differential game
$\Omega_x(t) \subseteq R^n$	state constraint
$S \subseteq R^n$	target set for the final state $x(t_b)$
$T(S, x) \subseteq R^n$	tangent cone of the target set S at x
$T^*(S, x) \subseteq R^n$	normal cone of the target set S at x
$T(\Omega, u) \subseteq R^m$	tangent cone of the constraint set Ω at u
$T^*(\Omega, u) \subseteq R^m$	normal cone of the constraint set Ω at u

Integers

i, j, k, ℓ	indices
m	dimension of the control vector
n	dimension of the state and the costate vector
p	dimension of an output vector
λ_0	scalar Lagrange multiplier for J, 1 in the regular case, 0 in a singular case

Functions

$f(.)$	function in a static optimization problem
$f(x,u,t)$	right-hand side of the state differential equation
$g(.)$, $G(.)$	define equality or inequality side-constraints
$h(.)$, $g(.)$	switching function for the control and offset function in a singular optimal control problem
$H(x,u,\lambda,\lambda_0,t)$	Hamiltonian function
$J(u)$	cost functional
$\mathcal{J}(x,t)$	optimal cost-to-go function
$L(x,u,t)$	integrand of the cost functional
$K(x,t_b)$	final state penalty term
$A(t)$, $B(t)$, $C(t)$, $D(t)$	system matrices of a linear time-varying system
F, $Q(t)$, $R(t)$, $N(t)$	penalty matrices in a quadratic cost functional
$G(t)$	state-feedback gain matrix
$K(t)$	solution of the matrix Riccati differential equation in an LQ regulator problem
$P(t)$	observer gain matrix
$Q(t)$, $R(t)$	noise intensity matrices in a stochastic system
$\Sigma(t)$	state error covariance matrix
$\kappa(.)$	support function of a set

Operators

$\frac{d}{dt}$, $\dot{}$	total derivative with respect to the time t
$\mathrm{E}\{\dots\}$	expectation operator
$[\dots]^{\mathrm{T}}$, T	taking the transpose of a matrix
U	adding a matrix to its transpose
$\dfrac{\partial f}{\partial x}$	Jacobi matrix of the vector function f with respect to the vector argument x
$\nabla_x L$	gradient of the scalar function L with respect to x, $\nabla_x L = \left(\dfrac{\partial L}{\partial x}\right)^{\mathrm{T}}$

1 Introduction

1.1 Problem Statements

In this book, we consider two kinds of dynamic optimization problems: optimal control problems and differential game problems.

In an optimal control problem for a dynamic system, the task is finding an admissible control trajectory $u : [t_a, t_b] \to \Omega \subseteq R^m$ generating the corresponding state trajectory $x : [t_a, t_b] \to R^n$ such that the cost functional $J(u)$ is minimized.

In a zero-sum differential game problem, one player chooses the admissible control trajectory $u : [t_a, t_b] \to \Omega_u \subseteq R^{m_u}$ and another player chooses the admissible control trajectory $v : [t_a, t_b] \to \Omega_v \subseteq R^{m_v}$. These choices generate the corresponding state trajectory $x : [t_a, t_b] \to R^n$. The player choosing u wants to minimize the cost functional $J(u, v)$, while the player choosing v wants to maximize the same cost functional.

1.1.1 The Optimal Control Problem

We only consider optimal control problems where the initial time t_a and the initial state $x(t_a) = x_a$ are specified. Hence, the most general optimal control problem can be formulated as follows:

Optimal Control Problem:
Find an admissible optimal control $u : [t_a, t_b] \to \Omega \subseteq R^m$ such that the dynamic system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t), t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

into an admissible final state

$$x(t_b) \in S \subseteq R^n \ ,$$

and such that the corresponding state trajectory $x(.)$ satisfies the state constraint

$$x(t) \in \Omega_x(t) \subseteq R^n$$

at all times $t \in [t_a, t_b]$, and such that the cost functional

$$J(u) = K(x(t_b), t_b) + \int_{t_a}^{t_b} L(x(t), u(t), t)\, dt$$

is minimized.

Remarks:

1) Depending upon the type of the optimal control problem, the final time t_b is fixed or free (i.e., to be optimized).
2) If there is a nontrivial control constraint (i.e., $\Omega \neq R^m$), the admissible set $\Omega \subset R^m$ is time-invariant, closed, and convex.
3) If there is a nontrivial state constraint (i.e., $\Omega_x(t) \neq R^n$), the admissible set $\Omega_x(t) \subset R^n$ is closed and convex at all times $t \in [t_a, t_b]$.
4) Differentiability: The functions f, K, and L are assumed to be at least once continuously differentiable with respect to all of their arguments.

1.1.2 The Differential Game Problem

We only consider zero-sum differential game problems, where the initial time t_a and the initial state $x(t_a) = x_a$ are specified and where there is no state constraint. Hence, the most general zero-sum differential game problem can be formulated as follows:

Differential Game Problem:
Find admissible optimal controls $u : [t_a, t_b] \to \Omega_u \subseteq R^{m_u}$ and $v : [t_a, t_b] \to \Omega_v \subseteq R^{m_v}$ such that the dynamic system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t), v(t), t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

to an admissible final state

$$x(t_b) \in S \subseteq R^n$$

and such that the cost functional

$$J(u) = K(x(t_b), t_b) + \int_{t_a}^{t_b} L(x(t), u(t), v(t), t)\, dt$$

is minimized with respect to u and maximized with respect to v.

Remarks:

1) Depending upon the type of the differential game problem, the final time t_b is fixed or free (i.e., to be optimized).
2) Depending upon the type of the differential game problem, it is specified whether the players are restricted to open-loop controls $u(t)$ and $v(t)$ or are allowed to use state-feedback controls $u(x(t), t)$ and $v(x(t), t)$.
3) If there are nontrivial control constraints, the admissible sets $\Omega_u \subset R^{m_u}$ and $\Omega_v \subset R^{m_v}$ are time-invariant, closed, and convex.
4) Differentiability: The functions f, K, and L are assumed to be at least once continuously differentiable with respect to all of their arguments.

1.2 Examples

In this section, several optimal control problems and differential game problems are sketched. The reader is encouraged to wonder about the following questions for each of the problems:

- Existence: Does the problem have an optimal solution?
- Uniqueness: Is the optimal solution unique?
- What are the main features of the optimal solution?
- Is it possible to obtain the optimal solution in the form of a state feedback control rather than as an open-loop control?

Problem 1: Time-optimal, friction-less, horizontal motion of a mass point

State variables:

$$x_1 = \text{position}$$
$$x_2 = \text{velocity}$$

control variable:

$$u = \text{acceleration}$$

subject to the constraint

$$u \in \Omega = [-a_{\max}, +a_{\max}] \ .$$

Find a piecewise continuous acceleration $u : [0, t_b] \to \Omega$, such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} s_a \\ v_a \end{bmatrix}$$