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In memory of Bruno Forte
mentor, teacher, collaborator, friend.

Preface

The idea of modeling the behaviour of phenomena at multiple scales has become a useful tool in both pure and applied mathematics. Fractal-based techniques lie at the heart of this area, as fractals are inherently multiscale objects. Fractals have increasingly become a useful tool in real applications; they very often describe such phenomena better than traditional mathematical models.

Fractal-Based Methods in Analysis draws together, for the first time in book form, methods and results from almost 20 years of research on this topic, including new viewpoints and results in many of the chapters. For each topic, the theoretical framework is carefully explained. Numerous examples and applications are presented.

The central themes are *self-similarity* across scales (exact or approximate) and *contractivity*. In applications, this involves introducing an appropriate space for contractive operators and approximating the "target" mathematical object by the fixed point of one of these contractions. Under fairly general conditions, this approximation can be extremely good. This idea emerged from *fractal image compression*, where an image is encoded by the parameters of a contractive transformation (see Sect. 3.1 and Figs. 3.3 and 3.4). The first step in extending this methodology is to construct interesting contractive operators on many different types of spaces. After this theoretical framework has been established, the next step is to apply the methodology in practical problems. In this book, we present extensive examples of both of these steps.

We originally conceived a document that we could give to our students to help them learn the background for their research. This document would contain an introduction to fractals via iterated function systems (IFSs) and some of the important subsequent developments, all from this IFS viewpoint. This document has since taken on a life of its own that has resulted in this book. The original goal is reflected in the second chapter, which is designed to serve as the basis for a course.

In the first chapter, we give a “bird’s-eye” overview of the area, painting with broad brushstrokes to give the reader the feel and philosophy of the approach. We touch on many of the topics and applications, hoping to share our amazement at the breadth of interesting mathematics and applications and to entice the reader into learning more.

In Chapter 2, we present a brief course on the classical topics in the iterated function systems viewpoint on fractals. In order to help the reader who might be seeing the material for the first time, we have included many exercises in this chapter. This is in keeping with our own desire to use this particular chapter as the basis for a course on IFS fractals.

In Chapters 3–5 we carefully develop the IFS framework in a large variety of settings. In particular, in Chapter 3 we develop a theory of IFSs on various spaces of functions, including the interaction of IFSs with integral transforms and IFSs on wavelet spaces. IFSs on spaces of transforms have been used in mathematical imaging, with IFSs on Fourier transforms having applications in magnetic resonance imaging (MRI). Chapter 4 extends this to IFSs on multifunctions (set-valued functions) and measure-valued functions. Again, the primary motivation and application for this framework is in mathematical imaging. Chapter 5 proceeds to a careful construction of the framework in various spaces of measures, with many new results. We consider signed measures, vector measures, and multimeasures (set-valued measures). Furthermore, there is a discussion of “generalized” measures as dual objects to Lipschitz spaces. This is a very useful class of “measures” for many purposes.

In Chapter 6, we turn to another classical topic in IFS fractals, that of ergodic theory and the “chaos game.” An IFS defines a dynamical system, which in turn generates an invariant measure. Chapter 6 extends the “chaos game” to IFSs on various function spaces, including random algorithms for generating wavelet analysis and wavelet synthesis.

Chapters 7 and 8 present an extensive range of applications of fractal-based methods to inverse problems. The models that we discuss span the range from ordinary differential equations (ODEs), partial differential equations (PDEs), and random differential equations to stochastic differential equations (SDEs) and more. The range of application topics presented is equally broad, from models in physics to biological, population, and economic models. These two chapters are just the start of the possible application areas and serve to illustrate the power of fractal-based methods.

The preface of a book is also the customary place for acknowledgments and expressions of thanks to appropriate people – collaborators, students, and individuals who have, in whatever way, helped the authors with their work or understanding of the subject material. In our case, the list of such people is quite long and the risk of omission quite large, so we shall keep our acknowledgments rather brief. First of all (and in a somewhat chronological order), Ed Vrscay would like to thank Michael F. Barnsley who, while at Georgia Tech, introduced him to the fields of fractal geometry and fractal image compression. He would also like to thank Jacques Lévy-Véhel, Dietmar Saupe and Claude Tricot for invaluable discussions, collaborations and assistance that began in the late-1980s and led to the formation of the "Waterloo Fractal Coding and Analysis Project." It was the attraction of Bruno Forte, former Chair of Applied Mathematics, University of Waterloo, to the "Waterloo Project" that contributed to its significant initial growth, in particular the mathematical formulation of generalized fractal transforms and associated inverse problems. Further growth of the project was made possible with the arrival of Franklin Mendivil. After retiring from Waterloo in 1995, Bruno would return to Italy to assume Emeritus Professorships, first at the University of Bari and then at the University of Verona. Here, he would eventually supervise Davide La Torre's Master's thesis on inverse problem for fractal transforms. Davide La Torre would like to thank Vincenzo Capasso and Bruno Forte for addressing him in these topics and for the suggestions, the inspiration, and the support that they have given him during his academic career. Bruno Forte was the first professor Herb Kunze met in the classroom while and undergraduate at the University of Waterloo. A few years later, Herb worked repeatedly as Bruno's teaching assistant for the same advanced Calculus course. Herb Kunze wishes to acknowledge the influential impact that Bruno had on his early academic life.

For these reasons we are dedicating this book to him.

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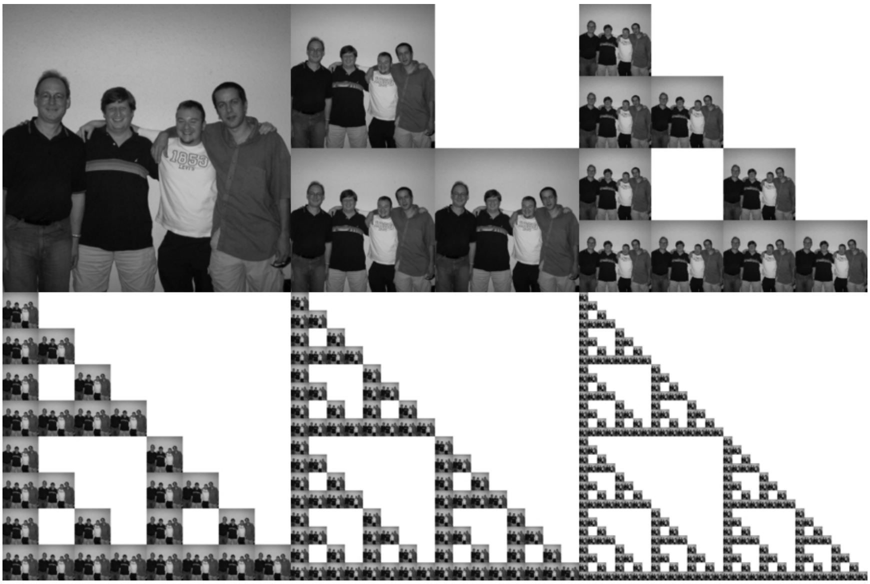

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