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Global Differential Geometry

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Global Differential Geometry

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Preface

This collection of research papers provides an overview over some of the progress that has been made in major areas in Differential Geometry in the past few years. It is centred around the scientific activities within the Priority Programme "Global Differential Geometry" supported by the German Research Foundation – Deutsche Forschungsgemeinschaft (DFG) – from 2003 until 2009. This Priority Programme, and hence the present volume, covers the following areas as well as their mutual connections:

- Global Riemannian Geometry
- Geometric Analysis
- Symplectic Geometry

In particular this volume offers the following topics:

The contributions to Global Riemannian Geometry include existence and obstruction results for metrics with particular properties, such as metrics under particular curvature and/or holonomy constraints, or metrics of low-dimensional geometries. Interesting aspects of geometric limits are also included. Some papers discuss asymptotic geometries, Euclidean buildings or singular spaces.

One of the topics in Geometric Analysis is the spectral geometry of elliptic operators on Riemannian manifolds, including their applications in differential topology. Another one is the geometry and analysis of Lorentzian manifolds, as well as classical and quantum fields on Lorentzian manifolds. Progress on mean curvature flow and scalar curvature constraints are also discussed.

Finally, the Symplectic Geometry section considers new aspects of Floer Homology and Contact Structures on odd-dimensional manifolds.

We hope this panoramic collection of papers will be helpful and inspiring.

Potsdam	*Christian Bär*
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Part I
Riemannian Geometry

Holonomy Groups and Algebras

Lorenz J. Schwachhöfer

1 Introduction

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its holonomy group which is defined as the subgroup $Hol_p(M) \subset \mathrm{Aut}(T_pM)$ consisting of all automorphisms of the tangent space T_pM at $p \in M$ induced by parallel translations along p-based loops.

The notion of *holonomy* first arose in classical mechanics at the end of the 19th century. It was Heinrich Hertz who used the terms *"holonomic" and "non-holonomic" constraints* in his magnum opus *Die Prinzipien der Mechanik, in neuen Zusammenhängen dargestellt* ("The principles of mechanics presented in a new form") which appeared one year after his death in 1895. For a more detailed exposition of the early origins of the holonomy problem, see also [21].

The notion of holonomy in the mathematical context seems to have appeared for the first time in the work of E.Cartan [30, 31, 33]. He considered the Levi-Civita connection of a Riemannian manifold M, so that the holonomy group is contained in the orthogonal group. He showed that in this case, the holonomy group is always connected if M is simply connected. Moreover, he observed that $Hol_p(M)$ and $Hol_q(M)$ are conjugate via parallel translation along any path from p to q, hence the holonomy group $Hol(M) \subset Gl(n, \mathbb{R})$ is well defined up to conjugation.

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Cartan's interest in holonomy groups was due to his observation that for a *Riemannian symmetric space*, the holonomy group and the isotropy group coincide up to connected components, as long as the symmetric space contains no Euclidean factor. This insight he used to classify Riemannian symmetric spaces [32].

In the 1950s, the concept of holonomy groups was treated more thoroughly. In 1952, Borel and Lichnerowicz [16] proved that the holonomy group of a Riemannian manifold is always a Lie subgroup, possibly with infinitely many components. In the same year, de Rham [37] proved what is nowadays called the *de Rham Splitting theorem*. Namely, if the holonomy of a Riemannian manifold is reducible, then the metric must be a local product metric; if the manifold is in addition complete and simply connected, then it must be a Riemannian product globally. In 1954, Ambrose and Singer proved a result relating the Lie algebra of the holonomy group and the curvature map of the connection [2].

A further milestone was reached by M.Berger in his doctoral thesis [9]. Based on the theorem of Ambrose and Singer, he established necessary conditions for a Lie algebra $\mathfrak{g} \subset \mathrm{End}(V)$ to be the Lie algebra of the holonomy group of a torsion free connection, and used it to classify all irreducible non-symmetric holonomy algebras of Riemannian metrics, i.e., such that $\mathfrak{g} \subset \mathfrak{so}(n)$. This list is remarkably short. In fact, it is included in (and almost coincides with) the list of connected linear groups acting transitively on the unit sphere. This fact was proven later directly by J.Simons [66] in an algebraic way. Recently, C.Olmos gave a beautiful simple argument showing this transitivity using elementary arguments from submanifold theory only [59].

Together with his list of possible *Riemannian* holonomy groups, Berger also gave a list of possible irreducible holonomy groups of *pseudo-Riemannian manifolds*, i.e., manifolds with a non-degenerate metric which is not necessarily positive definite. Furthermore, in 1957 he generalized Cartan's classification of Riemannian symmetric spaces to the isotropy irreducible ones [10].

In the beginning, it was not clear at all if the entries on Berger's list occur as the holonomy group of a Riemannian manifold. In fact, it took several decades until the last remaining cases were shown to occur by Bryant [16]. As it turns out, the geometry of manifolds with special holonomy groups are of utmost importance in many areas of differential geometry, algebraic geometry and mathematical physics, in particular in string theory. It would lead too far to explain all of these here, but rather we refer the reader to [11] for an overview of the geometric significance of these holonomies.

In 1998, S.Merkulov and this author classified all *irreducible* holonomy groups of torsion free connections [69]. In the course of this classification, some new holonomies were discovered which are *symplectic*, i.e., they are defined on a symplectic manifold such that the symplectic form is parallel. The first such symplectic example was found by Bryant [17]; later, in [34, 35] an infinite family of such connections was given. These symplectic holonomies share some striking rigidity properties which later were explained on a more conceptual level by M.Cahen and this author [26], linking them to parabolic contact geometry.

In this article, we shall put the main emphasis on the investigation of connections on principal bundles as all other connections can be deduced from these. This allows us to prove most of the basic results in greater generality than they were originally stated and proven. Thus, Sect. 2 is devoted to the collection of the basic definitions and statements, where in most cases, sketches of the proofs are provided. In Sect. 3, we shall collect the known classification results where we do not say much about the proofs, and finally, in Sect. 4 we shall describe the link of special symplectic connections with parabolic contact geometry.

2 Basic Definitions and Results

2.1 Connections on Principal Bundles

Let $\pi : P \to M$ be a (right)-principal G-bundle, where M is a connected manifold and G is a Lie group with Lie algebra $\mathfrak{g}$. A *principal connection on P* may be defined as a $\mathfrak{g}$- valued one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ such that:

1. ω is G-equivariant, i.e., $r^*_{g^{-1}}(\omega) = Ad_g \circ \omega$ for all $g \in G$,
2. $\omega(\xi^*) = \xi$ for all $\xi \in \mathfrak{g}$, where $\xi^*_p := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(p \cdot \exp(t\xi))$ denotes the action field corresponding to ξ.

Here, $r_g : P \to P$ denotes the right action of G. Alternatively, we may define a principal connection to be a G-invariant splitting of the tangent bundle

$$TP = \mathscr{H} \oplus \mathscr{V}, \text{ where } \mathscr{V}_p = \ker(d\pi)_p = span(\{\xi^*_p \mid \xi \in \mathfrak{g}\}) \text{ for all } p \in P. \quad (1)$$

In this case, $\mathscr{H}$ and $\mathscr{V}$ are called the *vertical* and *horizontal space*, respectively.

To see that these two definitions are indeed equivalent, note that for a given connection one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, we may define $\mathscr{H} := \ker(\omega)$; conversely, given the splitting (1), we define ω by $\omega|_{\mathscr{H}} \equiv 0$ and $\omega(\xi^*) = \xi$ for all $\xi \in \mathfrak{g}$; it is straightforward to verify that this establishes indeed a one-to-one correspondence.

The *curvature form* of a principal connection is defined as

$$\Omega := \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P) \otimes \mathfrak{g}. \quad (2)$$

For its exterior derivative we get

$$\mathrm{d}\Omega + [\omega, \Omega] = 0. \quad (3)$$

By the *Maurer-Cartan equations*, it follows from (2) that

$$\xi^* \lrcorner \Omega = 0 \text{ for all } \xi \in \mathfrak{g}, \text{ and } \mathrm{d}r^*_g(\Omega) = Ad_g \circ \Omega. \quad (4)$$

A (piecewise smooth) curve $c : [a, b] \to P$ is called *horizontal* if $c'(t) \in \mathscr{H}_{c(t)}$ for all $t \in [a, b]$. Evidently, for every curve $\underline{c} : [a, b] \to M$ and $p \in \pi^{-1}(\underline{c}(a))$, there is a unique horizontal curve $c^p : [a, b] \to P$, called *horizontal lift of* c, with $\underline{c} = \pi \circ c^p$ and $c^p(a) = p$. Since by the G-equivariance of $\mathscr{H}$ we have $c^{p\cdot g} = r_g \circ c^p$, the correspondence

$$\Pi_{\underline{c}} : \pi^{-1}(\underline{c}(a)) \longrightarrow \pi^{-1}(\underline{c}(b)), \qquad p \longmapsto c^p(b)$$

is G-equivariant and is called *parallel translation along* $\underline{c}$. The *holonomy at* $p \in P$ is then defined as

$$Hol_p := \{g \in G \mid p \cdot g = \Pi_{\underline{c}}(p) \text{ for } \underline{c} : [a, b] \to M \text{ with } \underline{c}(a) = \underline{c}(b) = \pi(p)\} \subset G. \tag{5}$$

Evidently, $Hol_p \subset G$ is a subgroup as we can concatenate and invert loops. Also, the G- equivariance of $\mathscr{H}$ implies that

$$Hol_{p\cdot g} = g^{-1}\, Hol_p\, g. \tag{6}$$

Moreover, if we pick any path $\underline{c} : [a, b] \to M$ then, again by concatenating paths, we obtain for $p \in \pi^{-1}(\underline{c}(a))$

$$Hol_{\Pi_{\underline{c}}(p)} = Hol_p. \tag{7}$$

Thus, by (6) and (7) it follows that the holonomy group $Hol \cong Hol_p \subset G$ is well defined up to conjugation in G, independent of the choice of $p \in P$.

We define the equivalence relation $\sim$ on P by saying that

$$p \sim q \text{ if } p \text{ and } q \text{ can be joined by a horizontal path.} \tag{8}$$

Then definition (5) can be equivalently formulated as

$$Hol_p := \{g \in G \mid p \cdot g \sim p\}. \tag{9}$$

Theorem 2.1. (Ambrose-Singer-Holonomy Theorem [2]). *Let $\pi : P \to M$ be a principal G-bundle with a connection $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ and the corresponding horizontal distribution $\mathscr{H} \subset TP$.*

1. The smallest involutive distribution on P which contains $\mathscr{H}$ is the distribution

$$\hat{\mathscr{H}}_p := \mathscr{H}_p \oplus \{\xi^*_p \mid \xi \in \mathfrak{hol}_p\},$$

where $\mathfrak{hol}_p \subset \mathfrak{g}$ is the Lie subalgebra generated by

$$\mathfrak{hol}_p = \langle\{\Omega(\mathrm{d}\Pi_{\underline{c}}(v), \mathrm{d}\Pi_{\underline{c}}(w)) \mid v, w \in T_pP, \underline{c} : [a,b] \\ \to M \textit{ any path with } \underline{c}(a) = \pi(p)\}\rangle. \tag{10}$$

2. *The identity component of* $(Hol_p)_0 \subset G$ *is a (possibly non-regular) Lie subgroup with Lie algebra* $\mathfrak{hol}_p$.

Proof. Observe first that the dimension of the right hand side of (10) is independent of $p \in P$. Indeed, from the definition, $\hat{\mathscr{H}}_{q\cdot g} = \mathrm{d}r_g(\hat{\mathscr{H}}_q)$, so that this dimension is independent of the point in the fiber of P; moreover, if $p \sim q$ and $\underline{c} : [a,b] \to M$ is a path with horizontal lift joining p and q, then it follows from the very definition that $\hat{\mathscr{H}}_q \cap \mathscr{V}_q = \mathrm{d}\Pi_{\underline{c}}(\hat{\mathscr{H}}_p \cap \mathscr{V}_p)$, and $\mathrm{d}\Pi_{\underline{c}}$ is an isomorphism.

To see that $\hat{\mathscr{H}}$ is involutive, let $\underline{X}, \underline{Y} \in \mathscr{X}(M)$ be vector fields and $X, Y \in \mathscr{X}(P)$ be their horizontal lifts. Note that the flows Φ^t_X and $\Phi^t_{\underline{X}}$ relate as

$$\Phi^t_X = \Pi_{\underline{c}^t_X}, \quad \text{where } \underline{c}^t_X : [0,t] \to M \text{ is a trajectory of } \underline{X}.$$

Therefore, if we let $\hat{\mathscr{V}}_p := \{\xi^*_p \mid \xi \in \mathfrak{hol}_p\}$, then the definition of $\mathfrak{hol}_p$ implies that $\Phi^t_X(\hat{\mathscr{V}}_p) = \hat{\mathscr{V}}_q$, where $q = \Phi^t_X(p)$ and thus, $[X, \hat{\mathscr{V}}_p] \subset \hat{\mathscr{V}}_p$ for all horizontal vector fields X, i.e., $[\mathscr{H}, \hat{\mathscr{V}}] \subset \hat{\mathscr{H}}$.

Next, by (2), $[X,Y] = -\xi^*_{\Omega(X,Y)}$ mod $\mathscr{H}$ for all horizontal vector fields X, Y so that $[\mathscr{H}, \mathscr{H}] \subset \hat{\mathscr{H}}$; finally, $[\hat{\mathscr{V}}, \hat{\mathscr{V}}] \subset \hat{\mathscr{V}}$ as $\mathfrak{hol}_p$ is a Lie algebra by definition.

Thus, $\hat{\mathscr{H}} \subset P$ is an involutive distribution. Conversely, the above arguments show that any involutive distribution containing $\mathscr{H}$ also contains $\hat{\mathscr{H}}$, so that $\hat{\mathscr{H}}$ is minimal as asserted.

Let $P_0 \subset P$ be a maximal leaf of $\hat{\mathscr{H}}$, let $p_0 \in P_0$ and let

$$H := \{g \in G \mid p_0 \cdot g \in P_0\} \subset G.$$

Since $\mathscr{H}$ and hence $\hat{\mathscr{H}}$ is G-invariant, it follows that $H \subset G$ is a subgroup. In fact, $H \subset G$ is a (possibly non-regular) Lie subgroup since $H \cong P_0 \cap \pi^{-1}(\pi(p_0))$. In fact, the restriction $\pi : P_0 \to M$ is a principal H-bundle.

Standard arguments now show that P_0 is indeed a single equivalence class w.r.t. $\sim$, so that $H = Hol_{p_0}$ is a Lie subgroup of G with Lie algebra $\mathfrak{hol}_p$. See e.g. [5] for details. □

Definition 2.2. Let $P \to M$ be a principal G-bundle, and let $H \subset G$ be a (possibly non-regular) Lie subgroup of G. We call a (possibly non-regular) submanifold $P' \subset P$ an H*- reduction of* P if the restriction $\pi : P' \to M$ is a principal H-bundle.

In particular, a maximal leaf $P_0 \subset P$ of the distribution $\hat{\mathscr{H}}$ from Theorem 2.1 is called a *holonomy reduction of* P which is therefore a reduction with structure group $Hol \subset G$. We denote the restriction of ω, Ω and $\mathscr{H}$ to P_0 by the same symbols.