Contributions in Mathematical and Computational Sciences 7

Thanasis Bouganis
Otmar Venjakob *Editors*

Iwasawa Theory 2012

State of the Art and Recent Advances

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Contributions in Mathematical and Computational Sciences • Volume 7

Editors
Hans Georg Bock
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Editors

Iwasawa Theory 2012

State of the Art and Recent Advances

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Preface to the Series

Contributions to Mathematical and Computational Sciences

Mathematical theories and methods and effective computational algorithms are crucial in coping with the challenges arising in the sciences and in many areas of their application. New concepts and approaches are necessary in order to overcome the complexity barriers particularly created by nonlinearity, high-dimensionality, multiple scales and uncertainty. Combining advanced mathematical and computational methods and computer technology is an essential key to achieving progress, often even in purely theoretical research.

The term mathematical sciences refers to mathematics and its genuine sub-fields, as well as to scientific disciplines that are based on mathematical concepts and methods, including sub-fields of the natural and life sciences, the engineering and social sciences and recently also of the humanities. It is a major aim of this series to integrate the different sub-fields within mathematics and the computational sciences, and to build bridges to all academic disciplines, to industry and other fields of society, where mathematical and computational methods are necessary tools for progress. Fundamental and application-oriented research will be covered in proper balance.

The series will further offer contributions on areas at the frontier of research, providing both detailed information on topical research, as well as surveys of the state-of-the-art in a manner not usually possible in standard journal publications. Its volumes are intended to cover themes involving more than just a single "spectral line" of the rich spectrum of mathematical and computational research.

The Mathematics Center Heidelberg (MATCH) and the Interdisciplinary Center for Scientific Computing (IWR) with its Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences (HGS) are in charge of providing and preparing the material for publication. A substantial part of the material will be acquired in workshops and symposia organized by these institutions in topical areas of research. The resulting volumes should be more than just proceedings collecting papers submitted in advance. The exchange of information and the discussions during the meetings should also have a substantial influence on the contributions.

This series is a venture posing challenges to all partners involved. A unique style attracting a larger audience beyond the group of experts in the subject areas of specific volumes will have to be developed.

Springer Verlag deserves our special appreciation for its most efficient support in structuring and initiating this series.

Heidelberg, Germany

Hans Georg Bock
Willi Jäger
Otmar Venjakob

Preface

Iwasawa Theory is one of the most active fields of research in modern Number Theory. The great interest in Iwasawa Theory is reflected by the highly successful bi-annual series of international conferences, starting in 2004 in Besancon and continuing in Limoges, Irsee and Toronto with a scientific committee formed by John Coates, Ralph Greenberg, Cornelius Greither, Masato Kurihara, and Thong Nguyen Quang Do. The Iwasawa 2012 Conference, organized by Otmar Venjakob and Thanasis Bouganis, took place in Heidelberg (July 30–August 3) and drew in over 120 participants. It was supported by the Mathematics Center Heidelberg (MATCH) and by the European Research Council (ERC) Starting Grant IWASAWA awarded to Otmar Venjakob. This volume, Iwasawa Theory 2012 – State of the Art and Recent Advances, presents research and overview articles contributed by conference speakers and participants, as well as lecture notes from an introductory mini-course given by Chris Wultrich and Xin Wan and held the week before the conference.

One can argue that Iwasawa Theory has its roots in the early nineteenth century and in the work of Ernst Kummer (29 January 1810–14 May 1893), who studied the class number of the cyclotomic field $Q(\zeta_p)$, in his approach to prove Fermat's Last Theorem. Kummer not only provided a solution to the theorem for a large class of prime exponents, but also discovered a link between the p-divisibility of the class number of $Q(\zeta_p)$ and the values of the Riemann zeta function at the negative integers. This link between arithmetic expressions and special values of zeta functions, which was later refined in the work of Herbrand and Ribet, lies at the heart of modern number theory. It is the earliest example of a range of highly conjectural deep relations between arithmetic expressions and L-values, the most celebrated of which is the Conjecture of Birch and Swinnerton-Dyer.

However it was Kenkichi Iwasawa (September 11, 1917–October 26, 1998) and his Main Conjecture that completely transformed our view of the arithmetic of cyclotomic fields. Indeed Iwasawa, inspired by the work of Andre Weil on the Zeta Function of varieties over finite fields, initiated the systematic investigation of the p-part of the class number in the cyclotomic extension of Q. Not only did he manage to prove his deep theorems with respect to the growth of the p-part of the class number in such extensions; he also formulated his Main Conjecture, which relates the size of a particular Galois module to the Kubota-Leopold p-adic L-function. This conjecture would go on to serve as the prototype for an array of Main Conjectures, which predict a deep relation between p-adic L-functions and arithmetic invariants of abelian varieties or, even more generally, motives.

The Main Conjecture for cyclotomic fields is now a theorem, and considerable progress has also been made on other fronts, such as the Main Conjectures for CM fields, for elliptic modular forms and the Main Conjectures for abelian varieties over function fields. The proofs of all these Main Conjectures involve an impressive combination of various strands of pure mathematics such as K-theory, automorphic forms and algebraic geometry, contributing enormously to the popularity of the subject. Iwasawa Theory has not stopped growing in terms of its complexity and generalization. Undoubtedly the work of Hida, and his investigation of what are now referred to as Hida families, has transformed the way that we view Iwasawa Theory today. There has been also great interest in extending Iwasawa Theory to a non-abelian setting, where the focus is on the arithmetic behavior of the underlying motive over a p-adic Lie extension. A vast generalization of the Main Conjectures to this non-abelian setting has now been formulated and there have already been some first results, both in the number field and in the function field case.

It is exactly these astonishing new and rapid developments that the Iwasawa Conference series seeks to address. The main aim is to bring together experts from different strands in or closely related to Iwasawa Theory to report on recent developments and exchange ideas. The series has also established a tradition of very lively and pleasant meetings, a tradition that was strengthened by the 2012 conference. Events such as the half-day-long cruise on the Neckar River undoubtedly helped to create an inviting and stimulating atmosphere for the conference participants.

The week before the 2012 conference a preparatory mini-course was offered by Chris Wultrich and Xin Wan, aimed to introduce graduate students and newcomers to the field. While Chris Wultrich offered an overview of several basic aspects of Iwasawa Theory, Xin Wan provided an introduction to the work of Skinner and Urban on the Main Conjecture for elliptic modular forms. Their lecture notes, *Overview of Some Iwasawa Theory* by Chris Wultrich and *Introduction to Skinner-Urban's Work on the Main Conjecture for GL*2 by Xin Wan, are now presented as part of this volume. The organizers would like to take this opportunity to thank them again for their excellent lecture series in the summer of 2012 and their contributions to this volume.

The talks given in the Iwasawa 2012 conference covered the wide range of development in Iwasawa Theory over the past several years, and were complemented by a poster session. Not every contribution in this volume is based on a talk given during the conference; some of the contributions are survey articles, while others are original research articles appearing for first time in printed form.

Acknowledgements It is of course only the efforts of the contributors that made this volume possible, and the editors are grateful to them. Moreover the editors would like to thank the referees for their valued work, and to thank once again the speakers and the participants of the

2012 conference and the preparatory lecture series. Further the editors would like to express their gratitude to Birgit Schmoetten-Jonas for her support with organizing the 2012 conference and editing this volume, which has been nothing less than indispensable. Lastly, it is our pleasure to thank MATCH and ERC for their financial support, as well as Mrs. Allewelt and Dr. Peters from Springer Verlag for their excellent collaboration in editing this volume.

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Part I
Lecture Notes

Overview of Some Iwasawa Theory

Christian Wuthrich

1 Introduction

These are the notes to lectures given at Heidelberg in July 2012. The intention was to give an concise overview of some topics in Iwasawa theory to prepare the participants for the conference. As a consequence, they will contain a lot of definitions and results, but hardly any proofs and details. Especially I would like to emphasise that the word "proof" should be replaced by "sketch of proof" in all cases below. Also, I have no claim at making this a complete introduction to the subject, nor is the list of references at the end. For this the reader might find (Greenberg 2001) a better source.

The talks were given in four sessions, which form the four sections of these notes. We start by the classical Iwasawa theory for the class group, including the fundamental result of Iwasawa on the growth of class groups in $\mathbb{Z}_p$-extensions. We also describe Stickelberger elements, cyclotomic units and the main conjecture. This first section also contains the basic facts about Iwasawa algebras.

The second section introduces Iwasawa theory for elliptic curves by studying the growth of the Selmer group. We define Mazur-Stickelberger elements and the p-adic L-functions and state the main conjecture in this context. The third section includes the proof of the control theorem for Selmer groups (in the ordinary case) and the formula for the leading term of the characteristic series of the Selmer group. The last section shows how one generalises Selmer groups to various Galois representations. We conclude with a rough and short explanation about Kato's Euler system.

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