Simons Symposia

Werner Müller
Sug Woo Shin
Nicolas Templier *Editors*

Families of Automorphic Forms and the Trace Formula

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Editors
Werner Müller
Mathematical Institute
University of Bonn
Bonn, Nordrhein-Westfalen
Germany

Sug Woo Shin
Department of Mathematics
University of California, Berkeley
Berkeley, CA, USA

Nicolas Templier
Department of Mathematics
Cornell University
Ithaca, NY, USA

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Preface

The Simons symposium on families of automorphic forms and the trace formula took place in Puerto Rico from January 26th through February 1st of 2014. It was an opportunity to study families of automorphic representations of higher-rank groups with the goal of paving the way for future developments. We explored the trace formula, spectra of locally symmetric spaces, p-adic families, and other recent techniques from harmonic analysis and representation theory. Experts of different specialties discussed these topics together.

There were 23 participants. Background material has circulated in advance of the symposium, with the idea of focusing during the symposium on recent developments and conjectures toward the frontier of current knowledge. In addition to regular talks, open discussion sessions were scheduled daily for 1 h to promote in-depth exchanges. A different moderator was assigned to each session. The respective themes were: counting cohomological forms, p-adic trace formulas, Hecke fields, slopes of modular forms, and orbital integrals. The goal of each session was to isolate key difficulties and assess the feasibility of diverse approaches.

We hope that the activities of the symposium and the resulting 13 articles of this proceedings volume will be inspiring to participants and researchers in the field. Each article has been thoroughly refereed. Some articles contain original results that have not appeared before, some articles are a synthesis of current knowledge and future directions, and others are survey articles.

The symposium was made possible by the endeavor of the Simons Foundation which we would like to thank again for its generous support. We thank Yuri Tschinkel and Meghan Fazzi for their constant assistance in the organization. We thank the authors for contributing articles to these proceedings and also wish to thank the anonymous referees. Finally we thank Springer-Verlag for their help in publishing these proceedings.

Introduction

The symposium explored analytic, p-adic, and geometric perspectives on families of automorphic forms and the trace formula. An emphasis was on promoting the study of families on higher-rank groups, which was timely in view of recent spectacular developments in the Langlands program.

The Arthur–Selberg trace formula is one of the most important and fundamental tools in the theory of automorphic forms. Besides its indispensable role in reciprocity and functoriality, the trace formula is used to count automorphic forms and to globalize local representations to global automorphic forms, which has numerous applications. It continues to motivate a wide range of techniques in representation theory, in differential and algebraic geometry, and in analysis.

It has been a fruitful idea to study families when solving difficult problems, even if the problem concerns a single object. In the context of number theory, one can study an object, whether it is a variety, a representation, or an L-function, by deforming it in families. In deforming automorphic forms, harmonic families arise such as Dirichlet characters, holomorphic modular forms, Maass forms, Siegel modular forms, and automorphic representations with prescribed local components. The trace formula is essential in conceptualizing harmonic families and establishing their structural properties, such as the Sato–Tate equidistribution which generalizes the Weyl law, and limit multiplicities.

The study of families has taken a new turn in the last two decades with the advent of the Katz–Sarnak heuristics. For this and in other numerous applications of families to sieving, arithmetic statistics, zero-density estimates, L-values, diophantine equations, equidistribution of arithmetic cycles, the trace formula is again a key tool. Already in its most primitive version for GL(1) as the Poisson summation formula, it enters the theory of the distribution of prime numbers. This proceedings volume contributes to sharpening our knowledge of families and the trace formula with the expectation that it will drive new applications.

The trace formula is essential in the local Langlands correspondence and functoriality, starting from the work of Jacquet–Langlands and culminating in the work of Arthur on classical groups. For other applications, such as the ones mentioned above and many others, it is essential to allow a large class of test functions in order to get the most spectral information out of the trace formula. To this end, a number of deep problems in analysis need to be solved. On the spectral side, one has to deal with logarithmic derivatives of intertwining operators, which are the main ingredients of the terms associated to the Eisenstein series. On the geometric side, the singularities of orbital integrals play an important role, in addition to the volume terms which carry much of the arithmetic.

Toward the long-term goals of the subject, it is important to develop systematic ways to work with the local and global trace formulas, orbital integrals, trace characters, Plancherel measures, and other techniques from harmonic analysis and geometry. These themes have been developed separately over the years, and are now coming together.

As a quick guide for the reader, we give below a brief overview of each article in this volume and group them into the following four broad categories: geometric side, local representation theory, harmonic families, and p-adic families.

The geometric side of the trace formula has a rich arithmetic, algebraic, and combinatorial structure which has been studied for several decades. Arthur's fine expansion in weighted orbital integrals has opened the way to stabilization, endoscopic classification, and the fundamental lemma, which all have been achieved recently. Many more questions are now under investigation such as uniform expansions for test functions of non-compact support, a description of the global constants which are weighted generalizations of Tamagawa numbers, relations with the local trace formula, and analogues for function fields.

The article by Werner Hoffmann presents an approach to partition the geometric side according to a new equivalence relation which is finer than geometric conjugacy. The terms are then expressed in terms of certain prehomogeneous zeta functions. Supported by evidence coming from low-rank groups, several conjectures are stated with a view toward future developments.

The article by Jasmin Matz constructs a zeta function associated to the adjoint action of $\mathrm{GL}(n)$ on its Lie algebra. This zeta function is related to the Arthur–Selberg trace formula applied to certain non-compactly supported test functions. For $n = 2$ it coincides with Shintani's zeta function, and for $n = 3$ it is used to obtain results toward an asymptotic formula for the sum of residues of Dedekind zeta functions of families of real cubic fields.

Global orbital integrals factor as a product of local orbital integrals, thus generating interesting problems over local fields, Archimedean and non-Archimedean, of zero and positive characteristics. The solution of these problems involves a variety of techniques at the crossroad of harmonic analysis, algebraic geometry, and geometric representation theory.

The article by Jim Arthur develops a theory of germ expansions for weighted orbital integrals for real groups, thereby extending the pioneering work of Harish-Chandra in the unweighted case. These results will be useful for future investigations of invariant distributions and weighted orbital integrals, objects that are crucial in understanding the trace formula.

Motivic integration has its roots in quantifier elimination, resolution of singularities, and analytic continuation of Igusa integrals. It can be used to prove the transfer principle for the fundamental lemma, asserting that the matching of orbital integrals over a local field of equal characteristic is equivalent to the one over a local field of mixed characteristic. The article by Raf Cluckers, Julia Gordon, and Immanuel Halupczok concerns a related problem of uniform bounds for orbital integrals on p-adic groups as one varies the prime p, the conjugacy class, and the test function. One motivation comes from establishing the Sato–Tate equidistribution for families.

The spectral side of the trace formula consists of characters and weighted characters, which may be studied by local methods, and also terms of genuinely global nature, most notably the multiplicity of automorphic representations. Thanks to the works by Arthur, Moeglin-Waldspurger and others, we have the stabilization of the (twisted) trace formula, opening the doors for the full endoscopic classification of automorphic representations. Such a classification has been accomplished for quasi-split classical groups and anticipated for more groups in the near future. As a consequence, we have a deeper understanding of characters of reductive groups over local fields by relating characters of two different groups via endoscopic identities. In a different direction, the trace formula has been an indispensable tool in the study of asymptotic behavior of spectral invariants as exemplified by the Weyl law, the limit multiplicity problem, and more generally the Sato–Tate equidistribution for families. This allows another useful perspective on characters of reductive groups over local fields, e.g., by studying quantitative aspects of discrete series and formal degrees.

Tasho Kaletha surveys the new theory of local and global rigid inner forms, which seems indispensable in stating and proving a precise version of the Langlands correspondence and functoriality for reductive groups which are not quasi-split. The data for rigid inner forms are natural in that they determine a canonical normalization of transfer factors as well as the coefficients in the endoscopic character identities. The main advantage of Kaletha's approach over the previous ones is that every inner form over a local or global field admits at least one rigidification as a rigid inner form.

The article by Julee Kim, Sug Woo Shin, and Nicolas Templier studies an asymptotic behavior of supercuspidal characters of p-adic groups. The idea is that one can get a somewhat explicit control of the characters of supercuspidal representations constructed by Yu (which exhaust all supercuspidal representations if p is large by Kim's theorem). The main conjecture and its partial confirmation in the paper are motivated by an asymptotic study of the trace formula and analogy with Harish-Chandra theory of characters for real groups.

The Weyl law, the limit multiplicity problem and Sato-Tate equidistribution are some of the basic questions one can ask about the asymptotic distribution of automorphic forms. Originally, the Weyl law is concerned with the counting of eigenvalues of the Laplace operator on a compact Riemannian manifold. In the context of automorphic forms, it means that for a given reductive group we consider a family of cusp forms with fixed level and count them with respect to the analytic conductor. The goal is the same as above, namely, to establish an asymptotic formula for the number of cusp forms with fixed level and analytic contuctor bounded by a given number. Since in general, the underlying locally symmetric spaces are non-compact, it is much more subtle to establish the Weyl law in this setting. For GL(2) this problem was first approached by Selberg using his trace formula. In the higher-rank case, the Selberg trace formula is replaced by the Arthur trace formula.

The limit multiplicity problem is concerned with the limiting behavior of the discrete spectrum associated to congruence subgroups of a reductive group. For a given congruence subgroup of a reductive group G, one counts automorphic representations in the discrete spectrum whose Archimedean component belongs to a fixed bounded subset of the unitary dual of $G(\mathbb{R})$. The normalized counting function is a measure on the unitary dual, and the problem is to show that it approximates the Plancherel measure if the level of the congruence subgroup converges to infinity. This is known to be true for congruence subgroups of $\mathrm{GL}(n)$.

The first aim of the article by Peter Sarnak, Sug Woo Shin, and Nicolas Templier is to give a working definition for a family of automorphic representations. The definition given includes all known families. It distinguishes between harmonic families which can be approached by the trace formula and geometric families which arise from diophantine equations. One of the main issues is to put forth the basic structural properties of families. The implication is that one can define various invariants, notably the Frobenius–Schur indicator, moments of the Sato–Tate measure, a Sato–Tate group of the family, and the symmetry type. Altogether this refines the Katz–Sarnak heuristics and provides a framework for studying families and their numerous applications to sieving, equidistribution, L-functions, and other problems in number theory.

The article by Steven J. Miller et al. is a survey on results and works in progress on low-lying zeros of families of L-functions attached to geometric families of elliptic curves. The emphasis is on extended supports in the Katz–Sarnak heuristics and on lower-order terms and biases. The article begins with a detailed treatment of Dirichlet characters, which serves as an introduction to the techniques and general issues for a reader wishing to enter the subject.

The article by Werner Müller discusses the Weyl law and recent joint work with Finis and Lapid on limit multiplicities. Currently, both the geometric and the spectral sides can only be dealt with for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. Further research about the related problems is in progress. In the final section, the growth of analytic torsion is discussed. Analytic torsion is a sophisticated spectral invariant of an arithmetic group, whose growth with respect to the level aspect is related to the limit multiplicity problem and has consequences for the growth of torsion in the cohomology of arithmetic groups.

In her article on Hecke eigenvalues, Jasmin Matz discusses work concerning the asymptotic distribution of eigenvalues of Hecke operators on cusp forms for $\mathrm{GL}(n)$. Matz–Templier established the Sato–Tate equidistribution of Hecke eigenvalues for families of Hecke–Maass cusp forms on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. This has consequences for average estimates toward the Ramanujan conjecture and the distribution of low-lying zeros of each of the principal, symmetric square and exterior square L-functions. The Arthur–Selberg trace formula is used in the same way as in the case of the Weyl law.

A particular aspect of the limit multiplicity problem is the study of the growth of Betti numbers of congruence quotients of symmetric spaces if the level of the congruence subgroups tends to infinity. In his article, Simon Marshall establishes asymptotic upper bounds for the L^2-Betti numbers of the locally symmetric spaces associated to a quasi-split unitary group of degree 4, which improve the standard bounds. The main tool is the endoscopic classification of automorphic representations of quasi-split unitary groups by Mok.

Eigenvarieties and p-adic families of automorphic forms arose from the study of mod p and p-adic congruences of modular forms. They are the p-adic analogues of the harmonic families of automorphic forms in the context of the trace formula, but the p-adic version admits rigorous algebraic and geometric definitions and have been more thoroughly studied as such. Many analytic questions about families of automorphic forms can also be asked in the p-adic context. For instance the distribution of Hecke eigenvalues can be studied p-adically, and one could study families of p-adic L-functions instead of the usual L-functions. This could lead to novel and strong methods, especially if combined with the analytic approach.

Hida presents his results on the growth of Hecke fields in Hida families of Hilbert modular forms with motivation from Iwasawa theory. Hida's main theorem is that an irreducible component of the ordinary Hecke algebra is a CM-component, i.e., its associated Galois representation is dihedral, if and only if the Hecke field for that component has bounded degree over the p^{∞}-power cyclotomic extension over $\mathbb{Q}$ in some precise sense.

Buzzard and Gee introduce conjectures by Gouvêa, Gouvêa–Mazur, and Buzzard on the slopes of modular forms, namely, the p-adic valuations of the U_p-eigenvalues, for varying weights and fixed tame level. Despite computational evidence, the conjectures are largely open to date. The article points out a purely local phenomenon in the reduction of crystalline Galois representations motivated by the conjectures and proposes to make progress toward Buzzard's conjectures via modularity lifting theorems.

Bonn, Germany — Werner Müller
Berkeley, CA, USA — Sug Woo Shin
Ithaca, NY, USA — Nicolas Templier

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Germ Expansions for Real Groups

James Arthur

Abstract We shall introduce an archimedean analogue of the theory of p-adic Shalika germs. These are the objects for p-adic groups that govern the singularities of invariant orbital integrals. More generally, we shall formulate an archimedean theory of germs for weighted orbital integrals. In the process we shall be led to some interesting questions on a general class of asymptotic expansions. Weighted orbital integrals are the parabolic terms on the geometric side of the trace formula. An understanding of their singularities is important for the comparison of trace formulas. It might also play a role in the deeper spectral analysis of a single trace formula.

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1 Introduction

Suppose that G is a connected reductive group over a local field F of characteristic 0. The study of harmonic analysis on $G(F)$ leads directly to interesting functions with complicated singularities. If the field F is p-adic, there is an important qualitative description of the behaviour of these functions near a singular point. It is given by the Shalika germ expansion, and more generally, its noninvariant analogue. The purpose of this paper is to establish similar expansions in the archimedean case $F = \mathbb{R}$.

The functions in question are the invariant orbital integrals and their weighted generalizations. They are defined by integrating test functions $f \in C_c^\infty\big(G(F)\big)$ over strongly regular conjugacy classes in $G(F)$. We recall that $\gamma \in G(F)$ is strongly regular if its centralizer G_γ in G is a torus, and that the set G_{reg} of strongly regular

J. Arthur (✉)
Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, ON, Canada, M5S 2E4
e-mail: arthur@math.utoronto.ca

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elements is open and dense in G. If $\gamma \in G_{\mathrm{reg}}(F)$ approaches a singular point c, the corresponding orbital integrals blow up. It is important to study the resulting behaviour in terms of both γ and f.

The invariant orbital integral

$$f_G(\gamma) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx, \qquad \gamma \in G_{\mathrm{reg}}(F),$$

is attached to the invariant measure dx on the conjugacy class of γ. Invariant orbital integrals were introduced by Harish-Chandra. They play a critical role in his study of harmonic analysis on $G(F)$. The weighted orbital integral

$$J_M(\gamma,f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)dx, \qquad \gamma \in M(F) \cap G_{\mathrm{reg}}(F),$$

is defined by a noninvariant measure $v_M(x)dx$ on the class of γ. The factor $v_M(x)$ is the volume of a certain convex hull, which depends on both x and a Levi subgroup M of G. Weighted orbital integrals have an indirect bearing on harmonic analysis, but they are most significant in their role as terms in the general trace formula. In the special case that $M = G$, the definitions reduce to $v_G(x) = 1$ and $J_G(\gamma,f) = f_G(\gamma)$. Weighted orbital integrals therefore include invariant orbital integrals.

Suppose that c is an arbitrary semisimple element in $G(F)$. In Sect. 2, we shall introduce a vector space $\mathcal{D}_c(G)$ of distributions on $G(F)$. Let $\mathcal{U}_c(G)$ be the union of the set of conjugacy classes $\Gamma_c(G)$ in $G(F)$ whose semisimple part equals the conjugacy class of c. Then $\mathcal{D}_c(G)$ is defined to be the space of distributions that are invariant under conjugation by $G(F)$ and are supported on $\mathcal{U}_c(G)$. If F is p-adic, $\mathcal{D}_c(G)$ is finite dimensional. It has a basis composed of singular invariant orbital integrals

$$f \longrightarrow f_G(\rho), \qquad \rho \in \Gamma_c(G),$$

taken over the classes in $\Gamma_c(G)$. However if $F = \mathbb{R}$, the space $\mathcal{D}_c(G)$ is infinite dimensional. It contains normal derivatives of orbital integrals, as well as more general distributions associated with harmonic differential operators. In Sect. 2 (which like the rest of the paper pertains to the case $F = \mathbb{R}$), we shall describe a suitable basis $R_c(G)$ of $\mathcal{D}_c(G)$.

For p-adic F, the invariant orbital integral has a decomposition

$$f_G(\gamma) = \sum_{\rho\in\Gamma_c(G)} \rho^\vee(\gamma) f_G(\rho), \qquad f \in C_c^\infty\big(G(F)\big), \qquad (1)_p$$

into a finite linear combination of functions parametrized by conjugacy classes. This is the original expansion of Shalika. It holds for strongly regular points γ that are close to c, in a sense that depends on f. The terms

$$\rho^\vee(\gamma) = g_G^G(\gamma, \rho), \qquad \rho \in \Gamma_c(G),$$