Chapter 2
Oscillations and Fourier Analysis

2.1 Oscillatory Motion in Matter

A universal property of material objects is their ability to vibrate, whether the vibration results in an audible sound, as in the ringing of a bell, or is subtle and inaudible, as the motion in a quartz crystal. It can be a microscopic oscillation on an atomic scale, or as large as an earthquake. Oscillations in any part of an extended object or medium with undefined boundaries almost always propagate as waves.

If any solid object is struck with a sharp blow at some point, vibrations spread throughout the body, and waves are set up in the surrounding medium. If the medium is air, and we are within hearing range, the waves fall on our eardrums and are perceived as a loud sound, whose quality experience teaches us to differentiate according to the kind of object and the way it was struck. Unless the shape of the body and the way it was struck satisfy very particular conditions, the sound produced will be far from a pure tone. The sounds produced by different objects are recognizably different; even if we play the *same* note on different musical instruments, the quality of the sound, or *timbre*, as musicians call it, is different. It is a remarkable fact, first fully appreciated by Alexander Graham Bell, that just from the rapidly fluctuating air pressure of a sound wave falling on our eardrums we are able to construct what we should call an "acoustic image." That is, we are able to sort out and recognize the various sources of sound whose pressure waves have combined to produce a net complex wave pattern falling on the eardrum. To really appreciate how remarkable this facility is, imagine that a microphone is used to convert the complex fluctuations of pressure into an electrical signal that is connected through appropriate circuits to an oscilloscope, and you watched these fluctuations on the screen. Now, without being allowed to hear the sounds, imagine trying to recognize, just from the complex pattern, a friend's voice, or even that it is a human voice at all.

The reason that oscillatory motion is so universally present stems from two fundamental properties of matter. First, objects as we normally find them are in

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stable equilibrium; that is, any change in their shape brings into play a force to restore the undisturbed shape. Second, all objects have *inertia*; that is, once a body or part of a body has been set in motion, it will tend to continue in that state, unless forces are impressed upon it to change its state; this is the well-known first law of motion of Newton. It follows that when, for example, an external force causes a momentary displacement from equilibrium, the restoring force arising from the body's inherent equilibrium will cause the affected part of the body not only to return to the undisturbed state, but, because of inertia, to overshoot in the other direction. This in turn evokes again a restoring force and an overshoot, and so on.

2.2 Simple Harmonic Motion

The simplest form of oscillatory motion is *simple harmonic motion*, as exemplified by the swinging of a pendulum. This will ensue whenever a physical system is displaced from stable equilibrium by a sufficiently small amount that the restoring force varies nearly linearly with the displacement. Thus a Taylor expansion of the energy U in terms of a small displacement ξ about the point of stable equilibrium yields the following:

$$U = U_0 + a_2\xi^2 + a_3\xi^3 + \ldots \qquad (a_2 > 0),$$ 2.1

and for sufficiently small ξ the restoring force $F = -dU/d\xi$ may be taken as linear in the displacement. It follows that the equation of motion is given by

$$\frac{d^2\xi}{dt^2} + \frac{2a_2}{m}\xi = 0 \qquad (a_2 > 0),$$ 2.2

which has the well-known periodic solution

$$\xi = \xi_0 \cos\left[\omega t + \phi_0\right]$$ 2.3

characterized by a unique (angular) *frequency* ω, *amplitude* ξ_0 and initial *phase* ϕ_0. In a useful graphical representation, the displacement ξ is the projection onto a fixed straight line of a radius vector ξ_0 rotating with constant angular velocity ω; the quantity $(\omega t + \phi_0)$ is then the angular position of the radius vector, giving the phase of the motion. Such a representation is a *phasor diagram*, illustrated in Figure 2.1.

As a corollary, or simply by rewriting the solution in exponential form, it follows that the motion is the sum of two phasors of equal length rotating in opposite directions, thus

$$\xi = \frac{\xi_0}{2}e^{+i(\omega t + \varphi_0)} + \frac{\xi_0}{2}e^{-i(\omega t + \varphi_0)}.$$ 2.4

In the assumed linear approximation of the equation of motion, if ξ_1 and ξ_2 are two solutions of the equation, then any linear combination $(a\xi_1 + b\xi_2)$, where a and b are constants, is also a solution.
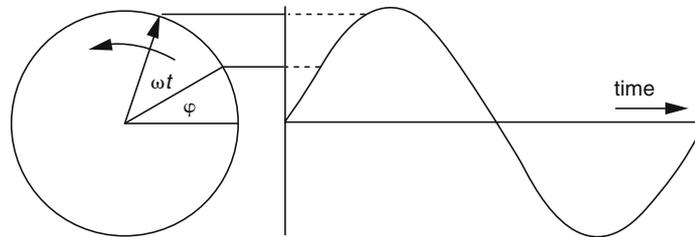
Figure 2.1 Simple harmonic motion as a projection of uniform circular motion: phasor diagram

If the next higher term in the expansion of U is retained, we are led to a non-linear (or *anharmonic*) oscillator. The prototypical example is the pendulum when the finite amplitude of oscillation is treated to a higher order of approximation than the simple linear one. Thus the exact equation of motion, expressed in terms of the angular deflection of the pendulum θ, is nonlinear, as follows:

$$l\frac{d^2\theta}{dt^2} + g\sin\theta = 0. \tag{2.5}$$

If $\theta \ll 1$, we may expand $\sin\theta$ in powers of θ to obtain a higher-order approximation to the equation of motion than the linear one. Thus

$$l\frac{d^2\theta}{dt^2} + g\left(\theta - \frac{1}{6}\theta^3\right) = 0. \tag{2.6}$$

Assume now that the amplitude of the motion is θ_0, so that in the linear approximation the solution would be $\theta_0\cos(\omega_0 t + \phi_0)$, where $\omega_0 = \sqrt{(g/l)}$. We can obtain an approximate correction to the frequency by using the method of successive approximation; this we do by assuming the following approximate form for the solution:

$$\theta = \theta_0\cos\omega t + \varepsilon\cos 3\omega t, \tag{2.7}$$

On substituting this into the equation of motion and setting the coefficients of $\cos\omega t$ and $\cos 3\omega t$ equal to zero, we find the following:

$$\omega = \omega_0\left(1 - \frac{\theta_0^2}{16}\right); \varepsilon = \frac{1}{3}\left(\frac{\theta_0}{4}\right)^3 (\theta_0 \ll 1), \tag{2.8}$$

which shows that the pendulum has a longer period at finite amplitudes than the limit as the amplitude approaches zero.

In the simple pendulum the suspended mass is constrained to move along the arc of a circle. It was this motion that Galileo thought to have the property of isochronism (or tautochronism), that is, requiring equal time to complete a cycle starting from any point on the arc. In fact, the mass must be constrained along a cycloid, the figure traced out by a point on a circle rolling on a straight line, rather

than a circle, in order to have this property. A more famous, related problem, one first suggested and solved by Bernoulli and independently by Newton and Leibnitz, has to do with the curve joining two fixed points along which the time to complete the motion is a *minimum* with respect to a variation in the curve; again the solution is a cycloid.

Attempts in the early development of pendulum clocks to realize in practice the isosynchronism of cycloidal motion were soon abandoned when it became apparent that *other* sources of error were more significant. In any event, in order to maintain a constant clock rate it is necessary only to regulate the amplitude of oscillation.

We should note that the presence of the nonlinear term in the equation of motion puts it in a whole different class of problems: those dealing with nonlinear phenomena. One far-reaching consequence of the nonlinearity is that the solution will now contain, in addition to the oscillatory term at the fundamental frequency ω, higher harmonics starting with 3ω. We will encounter in later chapters electronic devices of great practical importance whose characteristic response to applied electric fields is nonlinear.

2.3 Forced Oscillations: Resonance

Although our main concern will be the resonant response of atomic systems, requiring a quantum description, some of the basic classical concepts provide at least a background of ideas in which some of the terminology has its origins.

Imagine an oscillatory system, such as we have been discussing, having the further complication that its energy is slowly dissipated through some force resisting its motion. This is most simply introduced phenomenologically into the equation of motion as a term proportional to the time derivative of the displacement. The response of such a system to a periodic disturbance is governed by the following equation:

$$\frac{d^2\xi}{dt^2} + \gamma\frac{d\xi}{dt} + \omega_0^2\xi = \alpha_0 e^{ipt}, \qquad\qquad 2.9$$

which has the well-known solution

$$\xi = \frac{\alpha_0}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + \gamma^2 p^2}} e^{i(pt-\phi)} + \xi_0 e^{-\frac{\gamma}{2}l} e^{+i(\omega t + \Psi)}, \qquad\qquad 2.10$$

where $\phi = \arctan[\gamma p/(\omega_0^2 - p^2)]$ and $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$. The important feature of this solution is, of course, the resonantly large amplitude of the first term, the *particular integral*, at $\omega_0 = p$; but an equally significant point is that its *phase*, unlike that of the second natural oscillation term, bears a fixed relationship to that of the driving force. This means that if we have a large number of identical oscillators initially oscillating with random phases, and they are then subjected to the same driving force, the net global disturbance will simply be the sum of the resonant terms, since the other terms will tend to average out.

2.3.1 Response near Resonance: the Q-Factor

In order to analyze the behavior near resonance of a lightly damped oscillator for which $\gamma \ll \omega_0$, let us assume that $p = \omega_0 + \Delta$, where $\Delta \ll \omega_0$. Then we can write the following for the amplitude and phase of the impressed oscillation:

$$A = \frac{\alpha_0}{2\omega_0} \frac{1}{\sqrt{\Delta^2 + \left(\frac{\gamma}{2}\right)^2}}; \ \phi = \arctan\left(-\frac{\gamma}{2\Delta}\right), \ \Delta \ll \omega_0, \qquad 2.11$$

which, when plotted as functions of Δ, show for the amplitude the sharply peaked curve characteristic of resonance, falling to $1/\sqrt{2}$ of the maximum at $\Delta = -\gamma/2$ and $\Delta = +\gamma/2$, and for the phase, the sharp variation over that tuning range from $\pi/4$ to $3\pi/4$, passing through the value $\phi = \pi/2$ at exact resonance when $\Delta = 0$. A measure of the sharpness of the resonance, a figure of merit called the *Q-factor*, is defined as the ratio between the frequency and the resonance frequency width γ. Thus

$$Q = \frac{\omega_0}{\gamma}. \qquad 2.12$$

An equally useful result is obtained by relating Q to the rate of energy dissipation by the oscillating system. Thus from the equation of motion of the free oscillator we find after multiplying throughout by $d\xi/dt$ the following:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2}\omega^2\xi^2\right] = -\gamma\left(\frac{d\xi}{dt}\right)^2, \qquad 2.13$$

from which we obtain by averaging over many cycles (still assuming a weakly damped oscillator) the important result

$$\frac{d\langle U_{tot}\rangle}{dt} = -2\gamma\langle U_k\rangle; \ \langle U_k\rangle = \frac{1}{2}\langle U_{tot}\rangle, \qquad 2.14$$

From this follows the important result that we shall have many occasions to quote in the future:

$$Q = \omega_0 \frac{\langle U\rangle}{\frac{d\langle U\rangle}{dt}}. \qquad 2.15$$

Associated with the rapid change in amplitude is, as we have already indicated, a rapid change in the relative phase between the driving force and the response it causes. This interdependence between the amplitude and phase happens to be of particular importance in the classical model of *optical dispersion* in a medium as a manifestation of the resonant behavior of its constituent atoms to the oscillating electric field in the light wave.

   As we shall see in the next chapter, the sharp change in the phase ϕ as a function of frequency near resonance is of critical importance to the *frequency stability* of an oscillator, wherever the resonance is used as the primary frequency-selective element in the system. An important quantity from that point of view is the change
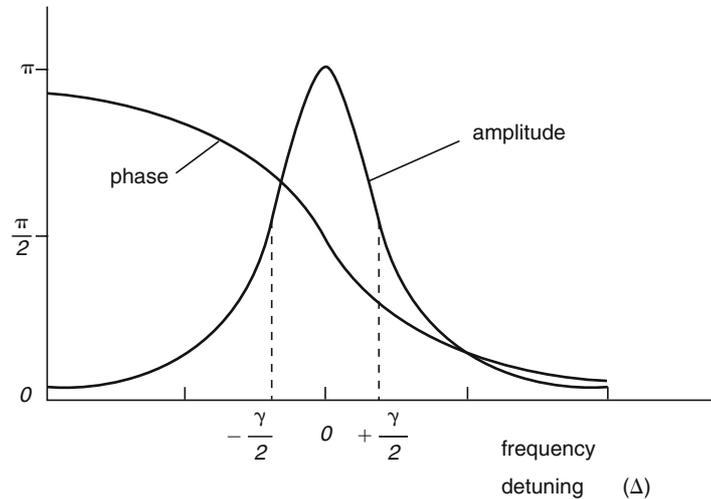
Figure 2.2 Amplitude and phase response curves versus frequency for a damped oscillator

in the phase angle produced by a given small detuning of the frequency from exact resonance. Figure 2.2 shows the approximate shapes of typical frequency-response curves. If we make the crude approximation that the phase varies linearly in the immediate vicinity of resonance, then since ϕ varies by π radians as the frequency is tuned from $-\gamma/2$ to $\gamma/2$, it follows that the change in phase $\Delta\phi$ is given approximately by the following:

$$\Delta\phi = \frac{(\omega_0 - \omega)}{\gamma}\pi. \hspace{3cm} 2.16$$

We note that having a very small γ, or equivalently, a very small fractional line width, favors a small change in frequency accompanying any given deviation in phase; and it is the phase that is susceptible to fluctuation in a real system.

2.4 Waves in Extended Media

In a region of space where a momentary disturbance takes place, whether among interacting material particles, as in an acoustic field, or charged particles in an electromagnetic field, such a disturbance generally propagates out as a wave. A historic example is the first successful effort to produce and detect electromagnetic waves as predicted by Maxwell's theory. Heinrich Hertz, at the University of Bonn, detected electromagnetic waves radiating from a "disturbance" in the form of a high-voltage spark. One of the physical conditions found in the propagation of a

disturbance as a wave is a delay in phase between the oscillation at a given point and that at an adjacent point along the direction of propagation; this is inevitably associated with a finite wave velocity.

The simplest case to analyze is that of transverse waves on a stretched string. It is evident in this case that the net force on a small element of the string depends on the *difference* between the directions of the string at the two ends of the given small segment and therefore depends on the *curvature* of the string. It follows by Newton's second law that the acceleration of this segment is proportional to the curvature; or stated symbolically, we have the well-known form of the (one-dimensional) *wave equation*:

$$T\frac{\partial^2 y}{\partial x^2} - \rho\frac{\partial^2 y}{\partial t^2} = 0, \tag{2.17}$$

where T and ρ are constants, the *tension* and *linear density* of the string. If we rewrite the equation as

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{2.18}$$

we can verify that a general solution, called *D'Alembert's solution*, can be written as follows:

$$y = f_1(x - Vt) + f_2(x + Vt), \tag{2.19}$$

where f_1 and f_2 are any (differentiable) functions, the first of which represents a disturbance traveling with a velocity V in the positive x direction, while the other is one traveling in the opposite direction, *without change of shape*: This is ultimately because V was assumed to be a constant.

In the case of the electromagnetic field, Maxwell's theory, the triumph of nineteenth-century physics, predicts that the electric and magnetic field vectors **E** and **B** propagate in a medium characterized by the electric permittivity ε and magnetic permeability μ according to the following wave equation expressed with reference to a Cartesian system of coordinates x, y, z:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \varepsilon\mu\frac{\partial^2 E_x}{\partial t^2} = 0, \tag{2.20}$$

with similar equations for the other components. It follows that for an unbounded uniform medium, the velocity of propagation $V = 1/\sqrt{\mu\varepsilon}$ is a constant, which in a vacuum has a numerical value in the MKS system of units of $2.9979\ldots\times10^8$ m/s.

The simplest solutions to the wave equation in an unbounded medium have a simple harmonic dependence on the coordinates and time, which in one dimension may be written in the form

$$E_y = E_0 \sin(kz - \omega t + \phi). \tag{2.21}$$

where k is the magnitude of the wave vector, ω is the (angular) frequency, and ϕ is an arbitrary phase.

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The surfaces of constant phase, defined by $(kz - \omega t)$ = constant, travel with the velocity V given by $V = \omega/k$. If we write, as is conventionally done, $V = c/n$ where c is the velocity of light *in vacuo,* then the quantity n, originally defined for frequencies in the optical range, is the refractive index that appears in Snell's law. This is the velocity of propagation only of the phase of a simple harmonic wave having a single frequency; for any more complicated wave, it becomes necessary to stipulate exactly what it is that the velocity refers to. Clearly, the concept of a wave velocity has meaning only if some identifiable attribute of the wave is indeed traveling with a well-defined velocity. If, for example, the wave has only one large crest like the bow wave of a ship traveling with sufficient speed, then the velocity with which that crest travels can differ from the phase velocity if the particular medium is *dispersive,* that is, if the phase velocity is a function of the frequency. This is readily seen if we recall that such a waveform can be thought of as a Fourier sum of simple harmonic waves, which now are assumed to travel at different velocities. In fact, there is no *a priori* reason for the wave to preserve its shape as it progresses; if it does not, the whole notion of wave velocity loses meaning. However, under some conditions a *group velocity* given by $V = d\omega/dk$ can be defined for a *wave packet.* More will be said about dispersive media in the next section.

It will be useful to review some of the fundamental properties of waves. Without going into great detail in the matter, we will simply state that at a boundary surface, where there is an abrupt change in the nature of the medium, waves will be partially reflected, and partially transmitted with generally a change in direction, that is, *refraction*, governed by Snell's law. The geometric surface joining all points that have the same phase is the *wavefront*, and in an unbounded medium the wavefront will advance at each point along the perpendicular to the surface, called a *ray* at that point.

If there is an obstruction in the medium, that is, a region where, for example, the energy of the wave is strongly absorbed, the waves will "bend around corners": the phenomenon of *diffraction*. This, it may be recalled, was the initial objection to the wave theory of light, an objection soon removed by the argument that the wavelength of light is extremely small compared with the dimensions of ordinary objects, and that diffraction is small under these conditions. The analysis of diffraction problems is based on *Huygens's principle*, as given exact mathematical expression by Kirchhoff, who showed that the solution to the wave equation at a given field point can be expressed as a surface integral of the field and its derivatives on a geometrical surface surrounding the field point. The evaluation of that surface integral is made tractable in the case of optical diffraction around large-scale objects by the smallness of the wavelength, which justifies a number of approximations. If an incident wave is delimited, for example by the aperture of an optical instrument or the antenna of a radio telescope, the field, of course, is nonzero only over the surface of the aperture, and the integral is simply over that surface. Application of the theory to the important case of a circular aperture under conditions referred to as *Frauenhoffer diffraction*, where the diffracted wave is brought to a

focus onto a plane, yields the following result for the intensity distribution in the focal plane:

$$I = 4I_0 \frac{J_1^2(ka\sin\theta)}{k^2 a^2 \sin^2\theta},$$

(2.22)

where a is the radius of the aperture and θ is the inclination of the direction of the field point with respect to the system axis. The Bessel function $J_1(ka\sin\theta)$ oscillates as the argument increases, implying an intensity pattern that consists of a central disk, called the *Airy disk*, surrounded by concentric bands that quickly fade as we go out from the center. Since the first zero of the Bessel function occurs when its argument is about 3.8, the radius of the Airy disk is therefore given by

$$\sin\theta \approx \theta \approx \frac{3.8}{ka} = 1.2\frac{\lambda}{D}.$$

(2.23)

In the approximation where *ray optics* are used, the image in the focal plane would of course have been a geometrical point.

2.5 Wave Dispersion

Another fundamental wave phenomenon is *dispersion*, the same phenomenon that was made manifest by Isaac Newton in his classic experiment on the dispersion of sunlight into its colored constituents using a glass prism. It occurs when the refractive index varies from one frequency to another; this can occur only in a material medium, never in vacuum, at least according to Maxwell's classical theory. The dispersive action of nonmagnetic dielectric materials is wholly due to the frequency dependence of the electric permittivity ε; this ultimately derives from the frequency dependence of the dynamical response of the molecular charges in the medium to the electric field component in the wave. This is a problem in quantum mechanics. However, H.A. Lorentz was able on the basis of his *electron theory* to account, at least qualitatively, for the gross features of the phenomenon. He assumed that the atomic particles exhibited resonant behavior at certain natural frequencies of oscillation and that the damping arises from interparticle collisions interrupting the phase of the particle oscillation.

According to this model, the oscillating electric field in the wave induces an oscillating polarization in each of the atomic particles with a definite phase relationship to the field, leading to a total global polarization, which for field vectors with the time dependence $\exp(-i\omega t)$ adds a resonant term to the permittivity, as follows:

$$\varepsilon = \left[1 + \frac{\sigma^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right]\varepsilon_0.$$

(2.24)

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where σ is a measure of the atomic oscillator strength. It follows that the (complex) refractive index n is given by

$$n = \frac{c}{V} = c \sqrt{\varepsilon_0 \mu_0 \left[1 + \frac{\sigma^2}{\omega_0^2 - \omega_0^2 - i\gamma\omega} \right]}, \qquad 2.25$$

from which we finally obtain, assuming that σ is small,

$$n = 1 + \frac{\sigma^2}{2} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} + i\frac{\sigma^2}{2} \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \qquad 2.26$$

Finally, substituting this result in the assumed (complex) form for the plane wave solution,

$$E_x = E_0 e^{i(nkz - \omega t)}, \qquad 2.27$$

we see that the real part of n determines the phase velocity and hence the dispersion, while the imaginary part yields an exponential attenuation of the wave amplitude, corresponding to absorption in the medium, provided that γ is a positive number. This shows explicitly how the real and imaginary parts of the atomic response determine the frequency dependence of the real and imaginary parts of the complex *propagation constant* through the medium, that is, of the refractive index and absorption of the wave. The complex propagation constant, as a function of frequency, exhibits a relationship between the real and imaginary parts that is an example of a far more general result that finds expression in what are called the Kramers–Kronig *dispersion relations*. It is far beyond the scope of this book to do more than mention that in a relativistic theory these relations are involved with the question of causality and the impossibility of a signal propagating faster than light.

2.6 Linear and Nonlinear Media

So far we have considered media that are *linear*, which means in the case of acoustic waves that a stress applied at some point produces a proportional strain; and conversely, a displacement from equilibrium brings about a proportional restoring force, resulting in simple harmonic motion. In the case of electromagnetic waves the classical theory leads to strictly linear equations *in vacuo*. A linear medium has an extremely important property: It obeys the *principle of superposition*. This states roughly that if more than one wave acts at a certain point, the resultant wave is simply the (vector) sum of these. At first, this may sound like pure tautology. The real meaning of the statement is that it is valid to talk about several waves being present simultaneously at a certain point as if they were individual entities that preserve their identity at the point where they overlap. A corollary is that in a linear medium, a wave is unchanged after it passes a region of overlap with another wave. According to classical theory, two light beams, no matter how powerful, intersecting in a vacuum will not interact with each other: each

emerges from the point of intersection as if the other beam were not there. In the realm of quantum field theory, however, it is another story: The vacuum state is far from "empty"!

However, it is possible to increase the strength of a disturbance in a *material* medium to such a point that the medium is no longer linear, and the principle of superposition no longer valid. Waves would then interact through the medium with each other, generating other waves at higher harmonic frequencies. We have already seen this in the case of the pendulum, where the presence of a nonlinear (third-degree) term in the equation of motion led to the presence of a third harmonic frequency.

In the more important circumstance where the field equations describing propagation through a given medium have a significant quadratic term, as in the frequency mixing devices we shall encounter later, two overlapping waves of frequencies ω_1 and ω_2 would interact, and the total solution would include the following:

$$\alpha\big[E_1(t) + E_2(t)\big]^2 = \alpha E_1^2 \cos^2(\omega_1 t) + \alpha E_2^2 \cos^2(\omega_2 t)$$
$$+ 2\alpha E_1 E_2 \cos(\omega_1 t)\cos(\omega_1 t) + \ldots \qquad 2.28$$

Using the trigonometric identities:

$$\cos^2(\omega t) = \frac{1}{2}\big[\cos(2\omega t) + 1\big],$$

$$\cos(\omega_1 t)\cos(\omega_2 t) = \frac{1}{2}\big[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t\big], \qquad 2.29$$

we see that with the assumed degree of nonlinearity, the second harmonic as well as the sum and difference frequencies appear in the output. By suitable filtering, any one of these frequency components can be isolated. We will have occasion to discuss in a later chapter the use of nonlinear crystal devices to produce intercombination and harmonic frequencies in the radio frequency and optical regions of the spectrum.

2.7 Normal Modes of Vibration

When waves are set up in a medium with a closed boundary surface, there will be reflections at different parts of the boundary, with the possibility of multiple reflections in which reflected waves are themselves reflected from opposing surfaces, all combining to produce a resultant wave pattern. If the medium is linear, the problem of finding the resultant is simply a matter of summing over the individual waves. It is one of the fundamental characteristics of waves that the resultant amplitude at a given point can be large or small depending on the relative phase of the combining waves at that point, producing an *interference pattern*.

Let us consider a homogeneous medium with a pair of parallel planes forming part of its boundary surfaces; the remainder of the boundary is immaterial. Let us

assume that a disturbance has been created at some point in this medium, giving rise to a wave that will travel out and be reflected by each of the plane boundary surfaces, return to the opposite surfaces, and be reflected again to pass through the initial point. The total distance traversed in making this round trip will be the same for all initial points and equal to twice the distance between the plane boundary surfaces. If this distance happens to be equal to a whole number of wavelengths of the wave, the waves arriving back at any initial point will, after an even number of reflections, be in phase with the initial disturbance, and the wave amplitude will build up at all points, as long as the external excitation continues. By contrast, if the round trip distance is not a whole number of wavelengths, the reflected waves will not be in phase with the exciting source, nor with waves from prior reflections, and the resultant of many even slightly out of phase waves will be weak and evanescent. Note that it is not necessary that the phase difference be near 180° to lead to cancellation and a weak resultant wave; even a small difference in phase produced in each round trip will accumulate after many successive reflections to result in the presence of waves having a phase ranging from 0° to 360°. In that event, for every wave of a given phase, there will be another wave 180° out of phase with it, leading to cancellation.

The condition for a buildup of the wave can be simply stated as follows:

$$2L = n\lambda_n, \hspace{4cm} 2.30$$

where n is any positive integer. This allows us to calculate the corresponding frequencies $\nu_n = V/\lambda_n = nV/2L$. Thus if we know the wave velocity V in the given medium and the distance between the reflecting surfaces, we can predict that certain frequencies of excitation will find a strong response, while any other, even neighboring, frequencies will not do so. Since n can be any whole number, there is an infinite number of frequencies forming a *discrete* spectrum, in which the frequencies have separate, isolated values, as opposed to a continuous spectrum in which frequency values can fall arbitrarily close to each other and merge into a continuum. The simplicity of the result, that the frequencies in the spectrum are whole multiples of the fundamental frequency $V/2L$, is due to the simple geometry of two plane reflecting surfaces in a homogeneous medium. However, even for more complicated geometries, part of the spectrum may still be discrete; but the frequencies will not necessarily be at equal increments.

To further elaborate on these basic concepts, let us consider another system, one that better lends itself to graphical illustration: a vibrating string stretched between two fixed points. Note that we can think of the fixed points merely as points where the string joins another string of infinite mass, and therefore we can regard the fixed points as the "boundaries" between two media. It has a discrete spectrum consisting of a *fundamental* frequency $\nu = V/2L$ and integral multiples of it called *harmonics*. In a musical context the harmonics above the first are called overtones, whose excitation determines the quality of the sound. These are the frequencies of the so-called *normal modes* of vibration of the string, shown in Figure 2.3.
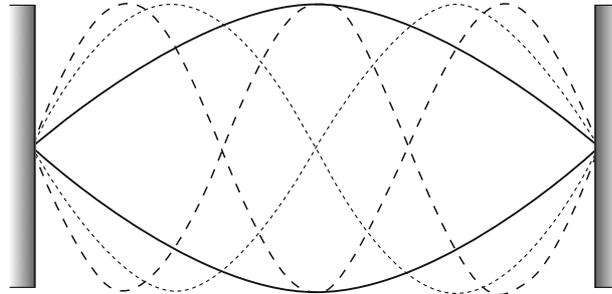
Figure 2.3 The natural modes of vibration of a stretched string

Each can be excited by applying an external periodic force, and the amplitude resulting from such excitation is qualitatively easy to predict: it is essentially zero unless the frequency is in the immediate neighborhood of one of the natural frequencies. At that point, the amplitude would grow indefinitely if it were not for frictional forces, or the onset of some amplitude-dependent mechanism to limit its growth. This phenomenon is of course *resonance*, which provides a method of determining the normal mode frequencies of oscillation of the system. At other frequencies the buildup of excitation is weak because of the mismatch in phase, as already described. Just how complete the cancellation will be depends on the highest number of reflections represented among the waves contributing to the resultant. It may be said approximately that for complete cancellation, the number of waves must be large enough that phase shifts spanning the entire 360° will be present. Now, the increment in phase per round trip is $360\,(\Delta\nu \cdot 2L/V)$ degrees, where $\Delta\nu$ is a small offset in frequency from one of the discrete frequencies in the spectrum. Thus for cancellation, we require a number N of traversals such that $N \cdot 360\,(\Delta\nu \cdot 2L/V) = 360$; that is, $\Delta\nu \cdot 2NL/V = 1$. But $2NL/V$ is simply the total time the wave has traveled back and forth, which in reality will be limited by internal frictional loss of energy in the string and imperfect reflections at the end points. Thus if we write $\Delta\tau$ for the mean time it takes the wave to become insignificant, then the smallest $\Delta\nu$ for cancellation is given by $\Delta\nu \cdot \Delta\tau \approx 1$; a smaller frequency offset gives only partial cancellation. This implies that in determining the frequency of resonance there is effectively a spread, or uncertainty, in the result if the measurement occupies a finite interval of time. This result, arrived at in a simple-minded way, hints at a much more general and fundamental result concerning uncertainties in the simultaneous observation of physical quantities: the now famous *Heisenberg Uncertainty Principle*. This principle applies to the simultaneous measurement of such quantities as frequency and time, which are said to be *complementary*, for which a determination of the frequency implies a finite time to accomplish it. Therefore, by its very nature, we cannot specify the frequency of an oscillation at a precise instant in time. To quantify this idea requires a precise definition of "uncertainty" in a physical measurement, which Heisenberg did in the context of quantum theory.

2.8 Parametric Excitations

The most often cited and certainly most dramatic example of the effects of resonance is the collapse of the suspension bridge across the Tacoma Narrows in Washington State, USA. Although its failure was due to violent oscillations, there was no external periodic force acting on it, but rather a buildup of what are called *parametric* oscillations, much like the fluttering of (venetian) window blinds in a steady wind. Such oscillations are characterized by a buildup resulting from some dynamic parameter varying in a particular way *within* each cycle.

There is another interesting phenomenon, in which a steady stream of air excites sound vibrations in a stretched string: the aeolian (from the Greek *aiolios*, wind) harp or lyre. This is a stringed instrument consisting of a set of strings of equal length stretched in a frame. When a steady air current passes over the strings, it emits a musical tone. The mechanism by which this occurs is rather subtle, as shown by the observed fact that the pitch of the tone does not seem to depend on the length or tension in the string, which would certainly be the case if it were simply a matter of the resonant frequencies being excited. It is observed, however, that if the resonant frequencies of the strings are made to equal the tone produced by the wind, the sound is greatly reinforced. The pitch of the sound depends on the velocity of the wind and the diameter of the string. According to Rayleigh, the great nineteenth-century physicist, noted for his theory of sound, the sound arises from vortices (eddies) in the air produced by the motion of air across the strings.

The simplest example of parametrically driven oscillations is the "pumping" of a child's swing, in which the child extends and retracts its legs, thereby varying the effective length of the suspension, during each swing. If we assume that a parameter that determines the frequency ω_0, in this case the length of the pendulum, is modulated harmonically at double the oscillation frequency, then the equation of motion will have the following form:

$$\frac{d^2\theta}{dt^2} + \omega_0^2\big[1 + \varepsilon\cos(2\omega_0 t)\big]\theta = 0. \qquad 2.31$$

If we assume $\varepsilon \ll 1$, then we can look for an approximate solution of the following form:

$$\theta = a(t)\cos\omega_0 t + b(t)\sin\omega_0 t. \qquad 2.32$$

where $a(t)$ and $b(t)$ vary negligibly during an oscillation. By substituting this form into the equation of motion, we find by setting the coefficients of $\cos\omega_0 t$ and $\sin\omega_0 t$ equal to zero, and neglecting higher harmonic frequencies, that the amplitudes $a(t)$ and $b(t)$ must satisfy the following equations:

$$\frac{da}{dt} + \frac{\varepsilon\omega_0}{4}b = 0,$$
$$\frac{db}{dt} + \frac{\varepsilon\omega_0}{4}a = 0, \qquad 2.33$$

from which we obtain finally the possible solution

$$a(t) = a_1 e^{+\frac{\varepsilon\omega_0}{4}t} + a_2 e^{-\frac{\varepsilon\omega_0}{4}t}, \qquad\qquad 2.34$$

with a similar result for $b(t)$. The presence of the first term, with the positive exponent, shows that the amplitude will grow exponentially. It is important to note (although our simple discussion does not deal with it) that the excitation of a *parametric resonance* will occur over a precise *range* of frequencies of modulation of the parameter; and further, that if the system is initially undisturbed, so that both θ and $d\theta/dt$ are initially zero, the system will not be excited into oscillation.

2.9 Fourier Analysis

When a system is subjected to a *simple periodic* disturbance, its response, in general, will be an oscillation at the frequency of that disturbance, superimposed on whatever free, natural oscillations were already present. As we have seen in the case of a simple physical system consisting of a vibrating string, a large resonant response is induced by a simple periodic force only at one of its natural frequencies. In general, however, when a violin string is excited into vibration, for example by plucking it, the shape of the string is a complicated function of time. We might imagine a high-speed movie camera recording this complex wave motion frame by frame. Predicting the motion of a system produced by an arbitrary initial displacement from its quiescent state is a fundamental problem of physics. The term "motion" used here is not restricted to movement in space; it could be, for example, the variation of temperature throughout a body as determined by the laws that govern the flow of heat.

Since any given natural frequency can effectively be excited only by an oscillatory force at that frequency, it is reasonable to assume that if the excitation is a complicated function of time, the response at the different natural frequencies somehow is representative of the "amount" of those frequencies in the excitation function. From this it seems plausible that to every given excitation function of time there corresponds a unique set of amplitudes (and phases) of the natural-mode responses. This would imply that any given excitation function of time can be regarded as a sum over a unique set of harmonic oscillations at the natural frequencies. In fact, this is given precise mathematical expression in the *Fourier expansion theorem*, one of the most useful theorems in physics, named for Joseph Fourier, a French mathematician who made a systematic study of what is now called Fourier analysis. It applies equally to the representation of an arbitrary initial *shape* of the string as a sum over a unique set of simple harmonic functions of position, making up the natural modes of vibration. This is of such importance to the understanding of what we shall encounter in succeeding chapters on atomic resonance that we must devote some effort to understanding it. The theorem proves that almost any periodic waveform, of whatever shape, can be expressed as the sum over a series of

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harmonic functions having amplitudes unique to the waveform, and it gives formulas for computing those amplitudes. In the context of high-fidelity audio systems the term "harmonic distortion" is familiar: It refers to the power in the second and higher harmonics of the given frequency being reproduced. This assumes that a distorted waveform can be unambiguously specified as consisting of a fundamental and harmonic components. The theorem is based on a special property called *orthogonality* of the functions describing the normal modes of vibration. The term means the property of being "perpendicular," as might be applied to two vectors; for functions, the test for this property is that the average of the product of the functions be zero, when taken over the appropriate interval. In that sense they are "uncorrelated." In the case of the normal mode functions of the vibrating string, $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$, where n and m are integers, their product averaged over the interval $0 < x < L$ is zero. Thus

$$\int_0^L \sin\left(n\pi\frac{x}{L}\right) \sin\left(m\pi\frac{x}{L}\right) dx = 0, \; n \neq m. \qquad 2.35$$

In general, for any given periodic function, that is, one satisfying $f(x) = f(x+2\pi)$, orthogonality allows the amplitudes of the harmonics in the following *Fourier series* expansion of the function to be determined:

$$f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x) + \cdots$$
$$+ b_1 \cos(x) + b_2 \cos(2x) + b_3 \cos(3x) + \cdots. \qquad 2.36$$

Thus by multiplying both sides of equation 2.36 by $\sin(nx)$ and integrating over the fundamental interval we immediately obtain the amplitude a_n. Thus

$$\int_0^{2\pi} \sin(nx) f(x) dx = \pi a_n, \qquad 2.37$$

with a similar result for the amplitudes b_n by replacing $\sin(nx)$ with $\cos(nx)$. We note that the amplitude is in a sense a measure of the extent to which the given function correlates with the harmonic mode function.

The theorem proves that by including higher and higher harmonics, the exact function can be represented as closely as we please. It follows that the amplitudes must decrease as we go to higher-order harmonics, so that a fair representation may be achieved with a finite number of harmonics. As an example, in Figure 2.4a is shown a periodic sawtooth waveform and beside it, in Figure 2.4b, are shown the amplitudes of the first few harmonics plotted against frequency to display the spectrum of the wave. The effect of a filter that removes all but the first three harmonics is shown in Figure 2.4c. We should note that to represent sharp changes in the waveform requires the inclusion of the higher harmonics in the sum.

It is clear from what has been said that for a plucked string, the extent to which each of the natural frequencies will be excited will depend first on the amplitude of each Fourier component in the initial displacement and second on the degree
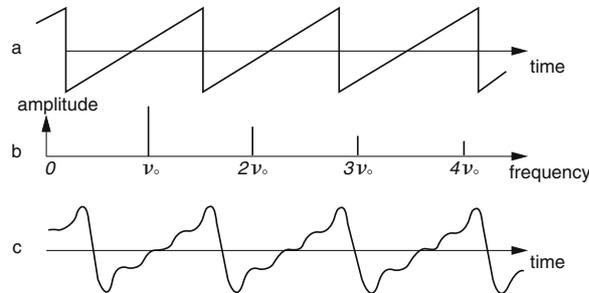
Figure 2.4 (a) A sawtooth waveform (b) its Fourier spectrum (c) the sum of the first three harmonics

to which each component is able to build up its amplitude in the presence of losses at the boundaries, etc. Since the initial amplitude of a given Fourier component according to the theorem is computed as an overlap integral between the given harmonic function and the function representing the initial displacement, the excitation of that particular harmonic is favored by having the initial displacement large where the harmonic displacement is large.

For non-periodic functions, there is a corresponding *Fourier integral theorem,* according to which, as a particular example, an even function $f(t)$ of time (that is, one satisfying $f(t) = f(-t)$) can be represented by the following integral:

$$f(t) = \int_0^\infty F(\omega) \cos(\omega t) d\omega, \qquad\qquad 2.38$$

where $F(\omega)$, now a function of a continuous variable, rather than the discrete mode index number n, gives the amplitude distribution over frequency, that is, the *Fourier spectrum* of the function $f(t)$. $F(\omega)$ has a unique, one-to-one relationship with $f(t)$, which the Fourier theorem proves is a reciprocal one, in the sense that $F(\omega)$ is obtained from $f(t)$ simply by interchanging their roles. The one function is called the *Fourier transform* of the other.

It frequently happens that where we have a complex signal consisting of what may appear as an unintelligible fluctuation in voltage, we are able to present the information in a far more useful way by applying the Fourier integral theorem. To show in a concrete way how this may be accomplished, consider the following hypothetical experiment. An input signal, which could, for example, be a sound wave or a microwave of complex waveform, is connected to an infinite number of ideal resonators tuned to progressively higher frequencies, with only a small increment in frequency between each resonator and its successor. This, it may be recalled, is the way it is thought that the human ear processes incoming sounds and is thereby able to separate the various types of sources that make up the complex waveform it receives. Let it be assumed that the input signal is switched on for a

predetermined period, after which it is switched off, and the amplitudes and phases of the oscillations in all the resonators are measured and then plotted against their resonant frequencies. Such a plot is the frequency spectrum of the incoming complex waveform, a waveform that begins as zero, jumps to the signal value when the switch is turned on, and goes back to zero when the switch is turned off. It is assumed that the frequency difference between consecutive resonators is small, so that there will be a very large number of them. The two principles that are the essence of this method of analysis are these: First, the phases and amplitudes of the resonators are unique to the incoming signal, and second, if we simply add simple harmonic oscillations at the frequencies of the resonators with those amplitudes and phases, the sum will reproduce the original signal waveform.

In later chapters we will have frequent occasion to refer to the Fourier spectra of signals. Two examples of Fourier transforms are shown in Figure 2.5. The first is a signal in which a simple oscillation is switched on at some point and thereafter slowly decays. The second represents a signal that really contains just one frequency, but the phase changes at irregular intervals of time.

In some important cases the phases are either indeterminate or inaccessible; in such cases the power spectrum, showing only the square of the amplitude at each frequency, is nevertheless very useful. The most obvious example is in the analysis of optical radiation, where of necessity we are limited to studying the power spectrum, since no common detector exists that can follow the extremely rapid oscillations in a light wave. Thus when sunlight, for example, is passed through a glass prism to separate the colors of the rainbow, as Newton did in his classic researches on the composition of white light, we are in a sense transforming the fluctuating field components in the incoming electromagnetic wave (the optical signal) into a continuous distribution of intensity over frequency, its Fourier spectrum. In this particular case, as blackbody radiation, the light from the sun will have phases that are random, which makes the availability of a representation in the form of a power spectrum, free of the phases, particularly crucial.
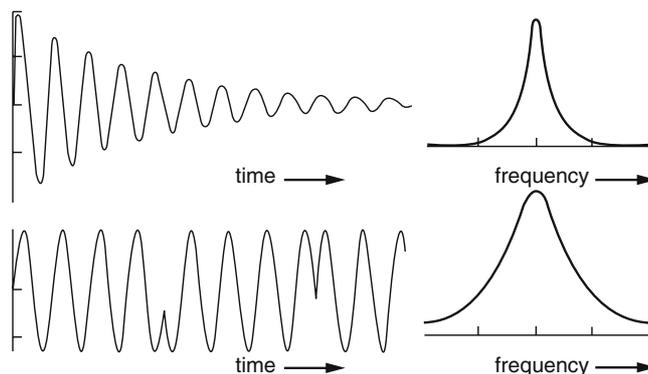

Figure 2.5 Examples of Fourier spectra

2.10 Coupled Oscillations

An important situation often arises in which one oscillatory system interacts with another. This often occurs where the oscillations of the one are to be synchronized with the other, a process familiar in television receivers. There the local sweep circuits that scan the picture have to be synchronized with the received horizontal and vertical synchronization pulses to obtain a stable picture. This, however, is synchronization under conditions in which the aspect we wish to consider is clearly absent: The oscillating systems do not interact directly. Let us consider, instead, two oscillating systems in which a resonant frequency in one nearly coincides with one in the other system, and assume that there is a weak coupling between them. A somewhat contrived example is shown in Figure 2.6, which depicts two pendulums (or is it pendula) of nearly equal natural oscillation period whose suspension is from a massive body that can slide horizontally without friction. If the coupling body were so massive that it may be regarded as immovable, then the pendulums would be independent of each other. However, for a large but finite mass, any oscillation in one pendulum affects the other. Perhaps the most striking phenomenon is seen in this system if we set one pendulum in motion while the other is left initially undisturbed. If we watch the subsequent motion of the two pendulums, a curious thing happens: The pendulum initially at rest will begin oscillating with increasing amplitude while the amplitude of the other simultaneously decreases. This will continue until the pendulum that was originally set in motion comes to rest, and the two have exchanged the initial state. Then the process reverses, and the two return to the original state. The energy of oscillation would continue to be exchanged back and forth indefinitely if it were not for the inevitable presence of frictional forces at the points of suspension and air resistance, which will cause the energy to be dissipated as heat and the system to come to rest. It is as if the system cannot "make up its mind" which state to be in; its oscillatory state is continually changing.
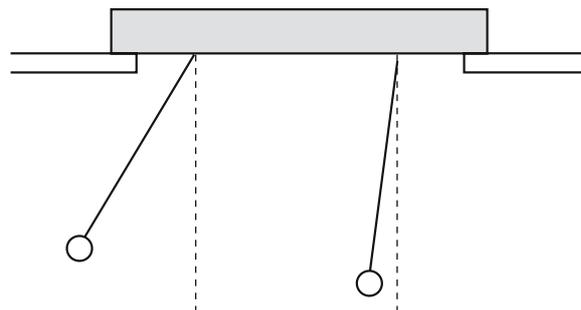

Figure 2.6 Two identical coupled pendula

An interesting question to ask about the coupled system is whether it can be set oscillating in some mode in which all parts of the system execute oscillation at the same frequency, with a stable amplitude. The attempt to answer this type of question, particularly to more complex systems involving several coupled systems, has led to a sophisticated theory and the concept of *normal* vibrations. To illustrate what is meant by the term, let us go back to the two coupled pendulums. We will state without proof that if this system is initially set in motion, either with the two pendulums in phase or the two exactly 180° out of phase, they will continue to oscillate in those modes with a constant amplitude. These two modes, illustrated in Figure 2.7, are called the normal modes of vibration for this particular system. It is important to note that for these modes to be preserved, the two pendulums must oscillate with a common frequency. This is, in fact, the defining characteristic of the normal modes: In a given mode, all parts of the coupled system must oscillate at one frequency belonging to that mode. The common frequency will, in general, vary from one mode to another. In the case of the two coupled pendulums, the frequencies of the two modes differ to an extent determined by the degree of coupling between them; this can be shown to be m/M, where m is the mass of the pendulum bob and M is the coupling mass. In terms of this coupling parameter m/M, the frequencies of the modes are approximately $\nu_a = \nu_o(1 + m/M)$ and $\nu_b = \nu_o$, where ν_o is the frequency of free oscillation in the absence of coupling.

It is interesting to view the original bizarre behavior, in which the oscillation went back and forth between the pendulums, in terms of the normal modes. We see that when only one pendulum is set in motion, the system is not in a normal mode but could be looked on as a "mixture" (or more precisely, a *linear superposition*) of the two normal modes; that is, the motion of each pendulum in our particular example is the *sum* of equal amplitudes of the two normal modes. But these modes do not have exactly the same frequency, and their relative phase will continuously increase, passing periodically through times when their phases differ by a whole number of cycles and are in step, and times when they get 180° out of step. When they are in step, they reinforce each other and produce a large amplitude, while the opposite is true when they get out of step and cancel each other. Thus the amplitude of each pendulum alternately rises and falls periodically, a phenomenon called "beats," from the way it is manifested when two musical notes having
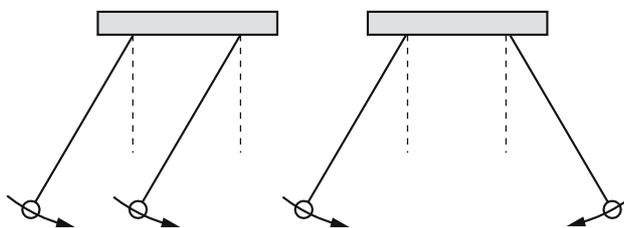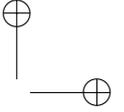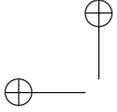

Figure 2.7 The normal modes of oscillation of two identical pendula

nearly the same pitch are sounded together. The number of beats per second can be shown to equal the difference between the frequencies of the two normal modes and therefore proportional to the strength of coupling, between the two pendulums; the tighter the coupling the higher the frequency at which the energy is exchanged back and forth between the two pendulums.