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Primes, Arithmetic Functions, and the Zeta Function

In this chapter we will discuss properties of primes and prime decomposition in the ring $A = \mathbb{F}[T]$. Much of this discussion will be facilitated by the use of the zeta function associated to A. This zeta function is an analogue of the classical zeta function which was first introduced by L. Euler and whose study was immeasurably enriched by the contributions of B. Riemann. In the case of polynomial rings the zeta function is a much simpler object and its use rapidly leads to a sharp version of the prime number theorem for polynomials without the need for any complicated analytic investigations. Later we will see that this situation is a bit deceptive. When we investigate arithmetic in more general function fields than $\mathbb{F}(T)$, the corresponding zeta function will turn out to be a much more subtle invariant.

Definition. The zeta function of A, denoted $\zeta_A(s)$, is defined by the infinite series

$$\zeta_A(s) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{1}{|f|^s}.$$

There are exactly q^d monic polynomials of degree d in A, so one has

$$\sum_{\deg(f) \leq d} |f|^{-s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \cdots + \frac{q^d}{q^{ds}},$$

and consequently

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}} \tag{1}$$

for all complex numbers s with $\Re(s) > 1$. In the classical case of the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, it is easy to show the defining series converges for $\Re(s) > 1$, but it is more difficult to provide an analytic continuation. Riemann showed that it can be analytically continued to a meromorphic function on the whole complex plane with the only pole being a simple pole of residue 1 at $s = 1$. Moreover, if $\Gamma(s)$ is the classical gamma function and $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, Riemann showed the functional equation $\xi(1-s) = \xi(s)$. What can be said about $\zeta_A(s)$?

By Equation 1 above, we see clearly that $\zeta_A(s)$, which is initially defined for $\Re(s) > 1$, can be continued to a meromorphic function on the whole complex plane with a simple pole at $s = 1$. A simple computation shows that the residue at $s = 1$ is $\frac{1}{\log(q)}$. Now define $\xi_A(s) = q^{-s}(1-q^{-s})^{-1}\zeta_A(s)$. It is easy to check that $\xi_A(1-s) = \xi_A(s)$ so that a functional equation holds in this situation as well. As opposed to case of the classical zeta-function, the proofs are very easy for $\zeta_A(s)$. Later we will consider generalizations of $\zeta_A(s)$ in the context of function fields over finite fields. Similar statements will hold, but the proofs will be more difficult and will be based on the Riemann-Roch theorem for algebraic curves.

Euler noted that the unique decomposition of integers into products of primes leads to the following identity for the Riemann zeta-function:

$$\zeta(s) = \prod_{\substack{p \text{ prime} \\ p>0}} (1 - \frac{1}{p^s})^{-1}.$$

This is valid for $\Re(s) > 1$. The exact same reasoning (which we won't repeat here) leads to the following identity:

$$\zeta_A(s) = \prod_{\substack{P \text{ irreducible} \\ P \text{ monic}}} (1 - \frac{1}{|P|^s})^{-1}. \tag{2}$$

This is also valid for all $\Re(s) > 1$.

One can immediately put Equation 2 to use. Suppose there were only finitely many irreducible polynomials in A. The right-hand side of the equation would then be defined at $s = 1$ and even have a non-zero value there. On the other hand, the left hand side has a pole at $s = 1$. This shows there are infinitely many irreducibles in A. One doesn't need the zeta-function to show this. Euclid's proof that there are infinitely many prime integers works equally well in polynomial rings. Suppose S is a finite set of irreducibles. Multiply the elements of S together and add one. The result is a polynomial of positive degree not divisible by any element of S. Thus, S cannot contain all irreducible polynomials. It follows, once more, that there are infinitely many irreducibles.

Let x be a real number and $\pi(x)$ be the number of positive prime numbers less than or equal to x. The classical prime number theorem states that

$\pi(x)$ is asymptotic to $x/\log(x)$. Let d be a positive integer and $x = q^d$. We will show that the number of monic irreducibles P such that $|P| = x$ is asymptotic to $x/\log_q(x)$ which is clearly in the spirit of the classical result.

Define a_d to be the number of monic irreducibles of degree d. Then, from Equation 2 we find

$$\zeta_A(s) = \prod_{d=1}^{\infty} (1 - q^{-ds})^{-a_d}.$$

If we recall that $\zeta_A(s) = 1/(1 - q^{1-s})$ and substitute $u = q^{-s}$ (note that $|u| < 1$ if and only if $\Re(s) > 1$) we obtain the identity

$$\frac{1}{1 - qu} = \prod_{d=1}^{\infty} (1 - u^d)^{-a_d}.$$

Taking the logarithmic derivative of both sides and multiplying the result by u yields

$$\frac{qu}{1 - qu} = \sum_{d=1}^{\infty} \frac{da_d\ u^d}{1 - u^d}.$$

Finally, expand both sides into power series using the geometric series and compare coefficients of u^n. The result is the beautiful formula,

Proposition 2.1.

$$\sum_{d|n} da_d = q^n.$$

This formula is often attributed to Richard Dedekind. It is interesting to note that it appears, with essentially the above proof, in a manuscript of C.F. Gauss (unpublished in his lifetime), "Die Lehre von den Resten." See Gauss [1], pages 608–611.

Corollary

$$a_n = \frac{1}{n} \sum_{d|n} \mu(d) q^{\frac{n}{d}}. \tag{3}$$

Proof. This formula follows by applying the Möbius inversion formula to the formula given in the proposition.

The formula in the above proposition can also be proven by means of the algebraic theory of finite fields. In fact, most books on abstract algebra contain the formula and the purely algebraic proof. The zeta-function approach has the advantage that the same method can be used to prove many other things as we shall see in this and later chapters.

The next task is to write a_n in a way which makes it easy to see how big it is. In Equation 3 the highest power of q that occurs is q^n and the next highest power that may occur is $q^{\frac{n}{2}}$ (this occurs if and only if $2|n$. All the other terms have the form $\pm q^m$ where $m \leq \frac{n}{3}$. The total number of terms is

$\sum_{d|n} |\mu(d)|$, which is easily seen to be 2^t, where t is the number of distinct prime divisors of n. Let $p_1, p_2, \ldots, p_t$ be the distinct primes dividing n. Then, $2^t \leq p_1 p_2 \ldots p_t \leq n$. Thus, we have the following estimate:

$$\left| a_n - \frac{q^n}{n} \right| \leq \frac{q^{\frac{n}{2}}}{n} + q^{\frac{n}{3}}.$$

Using the standard big O notation, we have proved the following theorem.

Theorem 2.2. (The prime number theorem for polynomials) *Let a_n denote the number of monic irreducible polynomials in $A = \mathbb{F}[T]$ of degree n. Then,*

$$a_n = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n} \right).$$

Note that if we set $x = q^n$ the right-hand side of this equation is $x/\log_q(x) + O(\sqrt{x}/\log_q(x))$ which looks like the conjectured precise form of the classical prime number theorem. This is still not proven. It depends on the truth of the Riemann hypothesis (which will be discussed later).

We now show how to use the zeta function for other counting problems. What is the number of square-free monics of degree n? Let this number be b_n. Consider the product

$$\prod_P (1 + \frac{1}{|P|^s}) = \sum \frac{\delta(f)}{|f|^s}. \tag{4}$$

As usual, the product is over all monic irreducibles P and the sum is over all monics f. We will maintain this convention unless otherwise stated. The function $\delta(f)$ is 1 when f is square-free, and 0 otherwise. This is an easy consequence of unique factorization in A and the definition of square-free. Making the substitution $u = q^{-s}$ once again, the right-hand side of Equation 4 becomes $\sum_{n=0}^{\infty} b_n u^n$. Consider the identity $1 + w = (1 - w^2)/(1 - w)$. If we substitute $w = |P|^{-s}$ and then take the product over all monic irreducibles P, we see that the left-hand side of Equation 4 is equal to $\zeta_A(s)/\zeta_A(2s) = (1 - q^{1-2s})/(1 - q^{1-s})$. Putting everything in terms of u leads to the identity

$$\frac{1 - qu^2}{1 - qu} = \sum_{n=0}^{\infty} b_n u^n.$$

Finally, expand the left-hand side in a geometric series and compare the coefficients of u^n on both sides. We have proven—

Proposition 2.3. *Let b_n be the number of square-free monics in A of degree n. Then $b_1 = q$ and for $n > 1$, $b_n = q^n(1 - q^{-1})$.*

It is amusing to compare this result with what is known to be true in $\mathbb{Z}$. If B_n is the number of positive square-free integers less than or equal to n, then $\lim_{n\to\infty} B_n/n = 6/\pi^2$. In less precise language, the probability that a positive integer is square-free is $6/\pi^2$. The probablity that a monic polynomial of degree n is square-free is b_n/q^n, and this equals $(1-q^{-1})$ for $n > 1$. Thus the probabilty that a monic polynomial in A is square-free is $(1-q^{-1})$. Now, $6/\pi^2 = 1/\zeta(2)$, so it is interesting to note that $(1-q^{-1}) = 1/\zeta_A(2)$. This is, of course, no accident and one can give good heuristic reasons why this should occur. The interested reader may want to find these reasons and to investigate the probablity that a polynomial be cube-free, fourth-power-free, etc.

Our next goal is to introduce analogues of some well-known number-theoretic functions and to discuss their properties. We have already introduced $\Phi(f)$. Let $\mu(f)$ be 0 if f is not square-free, and $(-1)^t$ if f is a constant times a product of t distinct monic irreducibles. This is the polynomial version of the Möbius function. Let $d(f)$ be the number of monic divisors of f and $\sigma(f) = \sum_{g|f} |g|$ where the sum is over all monic divisors of f.

These functions, like their classical counterparts, have the property of being multiplicative. More precisely, a complex valued function λ on $A-\{0\}$ is called multiplicative if $\lambda(fg) = \lambda(f)\lambda(g)$ whenever f and g are relatively prime. We assume λ is 1 on $\mathbb{F}^*$. Let

$$f = \alpha P_1^{e_1} P_2^{e_2} \dots P_t^{e_t}$$

be the prime decomposition of f. If λ is multiplicative,

$$\lambda(f) = \lambda(P_1^{e_1})\lambda(P_2^{e_2})\dots\lambda(P_t^{e_t}).$$

Thus, a multiplicative function is completely determined by its values on prime powers. Using multiplicativity, one can derive the following formulas for these functions.

Proposition 2.4. *Let the prime decomposition of f be given as above. Then,*

$$\begin{aligned}
\Phi(f) &= |f| \prod_{P|f} (1-|P|^{-1}), \\
d(f) &= (e_1+1)(e_2+1)\dots(e_t+1). \\
\sigma(f) &= \frac{|P_1|^{e_1+1}-1}{|P_1|-1} \cdot \frac{|P_2|^{e_2+1}-1}{|P_2|-1} \cdots \frac{|P_t|^{e_t+1}-1}{|P_t|-1}\ .
\end{aligned}$$

Proof. The formula for $\Phi(n)$ has already been given in Proposition 1.7.

If P is a monic irreducible, the only monic divisors of P^e are $1, P, P^2, \dots, P^e$ so $d(P^e) = e+1$ and the second formula follows.

By the above paragraph, $\sigma(P^e) = 1 + |P| + |P|^2 + \dots |P|^e = (|P|^{e+1} - 1/(|P|-1)$, and the formula for $\sigma(f)$ also follows.

As a final topic in this chapter we shall introduce the notion of the average values in the context of polynomials. Suppose $h(x)$ is a complex-valued function on $\mathbf{N}$, the set of positive integers. Suppose the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} h(n) = \alpha.$$

We then define α to be the average value of the function h. For example, suppose $h(n) = 1$ if n is square-free and 0 otherwise. Then, as noted above, the average value of h is known to be $6/\pi^2$. The sum $\sum_{k=1}^{n} h(k)$ sometimes grows too fast for the average value to exist. Often though, one can show the growth is dominated by a simple function of n. An example of this is the Euler ϕ-function. One can show

$$\sum_{k=1}^{n} \phi(k) = \frac{3}{\pi^2} n^2 + O(n \log(n)).$$

For this and other results of a similar nature, see Chapter VIII of the classic book by G.H. Hardy and E.M. Wright, Hardy and Wright [1]. Another good reference for this material is Chapter 3 of Apostol [1].

In the ring A the analogue of the positive integers is the set of monic polynomials. Let $h(x)$ be a function on the set of monic polynomials. For $n > 0$ we define

$$\mathrm{Ave}_n(h) = \frac{1}{q^n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} h(f).$$

This is clearly the average value of h on the set of monic polynomials of degree n. We define the average value of h to be $\lim_{n\to\infty} \mathrm{Ave}_n(h)$ provided this limit exists. This is the natural way in which average values arise in the context of polynomials. It is an exercise to show that if the average value exists in the sense just given, then it is also equal to the following limit:

$$\lim_{n\to\infty} \frac{1}{1 + q + q^2 \cdots + q^n} \sum_{\substack{f \text{ monic} \\ \deg(f)\le n}} h(f).$$

As we pointed out above, this limit does not always exist. However, even when it doesn't exist, one can speak of the average rate of growth of $h(f)$. Define $H(n)$ to equal the sum of $h(f)$ over all monic polynomials of degree n. As we will see, the function $H(n)$ sometimes behaves in a quite regular manner even though the values $h(f)$ vary erratically.

Instead of approaching these problems directly we use the method of Carlitz which uses Dirichlet series. Given a function h as above, we define the associated Dirichlet series to be

$$D_h(s) = \sum_{f \text{monic}} \frac{h(f)}{|f|^s} = \sum_{n=0}^{\infty} \frac{H(n)}{q^{ns}} \ . \tag{5}$$

In what follows, we will work in a formal manner with these series. If one wants to worry about convergence, it is useful to remark that if $|h(f)| =$

$O(|f|^\beta)$, then $D_h(s)$ converges for $\Re(s) > 1+\beta$. The proof just uses the comparison test and the fact that $\zeta_A(s)$ converges for $\Re(s) > 1$.

The right-hand side of 5 is simply $\sum_{n=0}^\infty H(n)u^n$, so the Dirichlet series in s becomes a power series in u whose coefficients are the averages $H(n)$. To see how this is useful, recall the function $d(f)$ which is the number of monic divisors of f. Let $D(n)$ be the sum of $d(f)$ over all monics of degree n (hopefully, this notation will not cause too much confusion). Then,

Proposition 2.5. $D_d(s) = \zeta_A(s)^2 = (1-qu)^{-2}$. *Consequently,* $D(n) = (n+1)q^n$.

Proof.

$$\zeta_A(s)^2 = \Big(\sum_h \frac{1}{|h|^s}\Big)\Big(\sum_g \frac{1}{|g|^s}\Big) =$$

$$\sum_f \Big(\sum_{\substack{h,g\\ hg=f}} 1\Big)\frac{1}{|f|^s} = \sum_f \frac{d(f)}{|f|^s} = D_d(s) \ .$$

This proves the first assertion. To prove the second assertion, notice

$$D_d(s) = \sum_{n=0}^\infty D(n)u^n = (1-qu)^{-2} \ .$$

It is easily seen that $(1-qu)^{-2} = \sum_{n=0}^\infty (n+1)q^n u^n$. Thus, the second assertion follows by comparing the coefficients of u^n on both sides of this identity.

A few remarks are in order. Notice that $\mathrm{Ave}_n(d) = n+1$ so the average value of $d(f)$ in the way we have defined it doesn't exist. On average, the number of divisors of f grows with the degree. If we set $x = q^n$ then our result reads $D(n) = x\log_q(x) + x$ which resembles closely the analogous result for the integers $\sum_{k=1}^n d(k) = x\log(x) + (2\gamma - 1)x + O(\sqrt{x})$ (here $\gamma \approx .577216$ is Euler's constant). This formula is due to Dirichlet. It is a famous problem in elementary number theory to find the best possible error term. In the polynomial case, there is no error term! This is because of the very simple nature of the zeta function $\zeta_A(s)$. Similar sums in the general function field context lead to more difficult problems. We shall have more to say in this direction in Chapter 17.

It is an interesting fact that many multiplicative functions have corresponding Dirichlet series which can be simply expressed in terms of the zeta function. We have just seen this for $d(f)$. More generallly, let $h(f)$ be multiplicative. The multiplicativity of $h(f)$ leads to the identity

$$D_h(s) = \prod_P \Big(\sum_{k=0}^\infty \frac{h(P^k)}{|P|^{ks}}\Big) \ .$$

As an example, consider the function $\mu(f)$. Since $\sum_{k=0}^{\infty} \frac{\mu(P^k)}{|P|^{ks}} = 1-|P|^{-s}$, we find $D_\mu(s) = \zeta_A(s)^{-1}$. The same method would enable us to determine the Dirichlet series for $\Phi(f)$ and $\sigma(f)$. However, we will follow a slightly different path to this goal.

Let λ and ρ be two complex valued functions on the monic polynomials. We define their Dirichlet product by the following formula (all polynomials involved are assumed to be monic)

$$(\lambda * \rho)(f) = \sum_{\substack{h,g \\ hg=f}} \lambda(h)\rho(g) \ .$$

This definition is exactly similar to the corresponding notion in elementary number theory. As is the case there, the Dirichlet product is closely related to multiplication of Dirichlet series.

Proposition 2.6.

$$D_\lambda(s)D_\rho(s) = D_{\lambda*\rho}(s) \ .$$

Proof. The calculation is just like that of Proposition 2.5.

$$D_\lambda(s)D_\rho(s) = \Big(\sum_h \frac{\lambda(h)}{|h|^s}\Big)\Big(\sum_g \frac{\rho(g)}{|g|^s}\Big) =$$

$$\sum_f \Big(\sum_{\substack{h,g \\ hg=f}} \lambda(h)\rho(g)\Big)\frac{1}{|f|^s} = D_{\lambda*\rho}(s) \ .$$

We now proceed to calculate the average value of $\Phi(f)$. We have seen that

$$\Phi(f) = |f| \prod_{P|f} (1-|P|^{-1}) \ .$$

Define $\lambda(f) = |f|$. A moment's reflection shows that the right hand side of the above equation can be rewritten as $\sum_{g|f} \mu(g)|f/g| = (\mu * \lambda)(f)$. Thus, by Proposition 2.6 we find

$$D_\Phi(s) = D_{\mu*\lambda}(s) = D_\mu(s)D_\lambda(s) = \zeta_A(s)^{-1}\zeta_A(s-1) \ . \qquad (6)$$

Proposition 2.7.

$$\sum_{\substack{\deg f=n \\ f \text{ monic}}} \Phi(f) = q^{2n}(1-q^{-1}) \ .$$

Proof. Let $A(n)$ be the left-hand side of the above equation. Then, with the usual transformation $u = q^{-s}$, Equation 6 becomes

$$\sum_{n=0}^{\infty} A(n)u^n = \frac{1-qu}{1-q^2u} \ .$$

Now, expand $(1 - q^2u)^{-1}$ into a power series using the geometric series, multiply out, and equate the coefficients of u^n on both sides. One finds $A(n) = q^{2n} - q^{2n-1}$. The result follows.

Finally, we want to do a similar analysis for the function $\sigma(f)$. Let $\mathbf{1}(f)$ denote the function which is identically equal to 1 on all monics f. For any complex valued function λ on monics, we see immediately that $(\mathbf{1} * \lambda)(f) = \sum_{g|f} \lambda(g)$. In particular, if $\lambda(f) = |f|$, then $(\mathbf{1} * \lambda)(f) = \sigma(f)$. Thus,

$$D_\sigma(s) = D_{\mathbf{1}*\lambda}(s) = D_{\mathbf{1}}(s) D_\lambda(s) = \zeta_A(s)\zeta_A(s-1) \ . \tag{7}$$

Proposition 2.8.

$$\sum_{\substack{\deg(f)=n \\ f \text{ monic}}} \sigma(f) = q^{2n} \cdot \frac{1 - q^{-n-1}}{1 - q^{-1}} \ .$$

Proof. Define $S(n)$ to be the sum on the left hand side of the above equation. Then, making the substitution $u = q^{-s}$ in Equation 7 we find

$$\sum_{n=0}^{\infty} S(n) u^n = (1 - qu)^{-1}(1 - q^2u)^{-1} \ .$$

Expanding the two terms on the right using the geometric series, multiplying out, and collecting terms, we deduce

$$S(n) = \sum_{k+l=n} q^k q^{2l} \ .$$

The result follows after applying a little algebra.

The method of obtaining average value results via the zeta function has now been amply demonstrated. The reader who wants to pursue this further can consult the original article of Carlitz [1]. Alternatively, it is an interesting exercise to look at Chapter VII of Hardy and Wright [1] or Chapter 3 of Apostol [1] , formulate the results given there for $\mathbb{Z}$ in the context of the polynomial ring $A = \mathbb{F}[T]$, and prove them by the methods developed above.

In Chapter 17, we will return to the subject of average value results, but in the broader context of global function fields.

Exercises

1. Let $f \in A$ be a polynomial of degree at least $m \geq 1$. For each $N \geq m$ show that the number of polynomials of degree N divisible by f divided by the number of polynomials of degree N is just $|f|^{-1}$. Thus, it makes sense to say that the probability that an arbitrary polynomial is divisible by f is $|f|^{-1}$.

2. Let $P_1, P_2, \ldots, P_t \in A$ be distinct monic irreducibles. Give a probabilistic argument that the probability that a polynomial not be divisible by any P_i^2 for $1 = 1, 2, \ldots, t$ is give by $\prod_{i=1}^{t}(1 - |P_i|^{-2})$.

3. Based on Exercise 2, give a heuristic argument to show that the probability that a polynomial in A is square-free is given by $\zeta_A(2)^{-1}$.

4. Generalize Exercise 3 to give a heuristic argument to show that the probability that a polynomial in A be k-th power free is given by $\zeta_A(k)^{-1}$.

5. Show $\sum |m|^{-1}$ diverges, where the sum is over all monic polynomials $m \in A$.

6. Use the fact that every monic m can be written uniquely in the form $m = m_0 m_1^2$ where m_0 and m_1 are monic and m_0 is square-free to show $\sum |m_0|^{-1}$ diverges where the sum is over all square-free monics m_0.

7. Use Exercise 6 to show

$$\prod_{\substack{P \text{ irreducible} \\ \deg P \leq d}} (1 + |P|^{-1}) \to \infty \quad \text{as} \quad d \to \infty \ .$$

8. Use the obvious inequality $1 + x \leq e^x$ and Exercise 7 to show $\sum |P|^{-1}$ diverges where the sum is over all monic irreducibles $P \in A$.

9. Use Theorem 2.2 to give another proof that $\sum |P|^{-1}$ diverges.

10. Suppose there were only finitely many monic irreducibles in A . Denote them by $\{P_1, P_2, \ldots, P_n\}$. Let $m = P_1 P_2 \ldots P_n$ be their product. Show $\Phi(m) = 1$ and derive a contradiction.

11. Suppose h is a complex valued function on monics in A and that the limit as n tends to infinity of $\mathrm{Ave}_n(h)$ is equal to α. Show

$$\lim_{n \to \infty} (1 + q + \cdots + q^n)^{-1} \sum_{\substack{f \text{ monic} \\ \deg f \leq n}} h(f) = \alpha \ .$$

12. Let $\mu(m)$ be the Möbius function on monic polynomials which we introduced in the text. Consider the sum $\sum_{\deg m = n} \mu(m)$ over monic polynomials of degree n. Show the value of this sum is 1 if $n = 0$, $-q$ if $n = 1$, and 0 if $n > 1$.

13. For each integer $k \geq 1$ define $\sigma_k(m) = \sum_{f|m} |f|^k$. Calculate $\mathrm{Ave}_n(\sigma_k)$.

14. Define $\Lambda(m)$ to be $\log|P|$ if $m = P^t$, a prime power, and zero otherwise. Show

$$\sum_{f|m} \Lambda(f) = \log|m| \ .$$

15. Show that

$$D_\Lambda(s) = -\zeta_A'(s)/\zeta_A(s).$$

Use this to evaluate $\sum_{\deg m = n} \Lambda(m)$.

16. Recall that $d(m)$ is the number of monic divisors of m. Show

$$\sum_{m \text{ monic}} \frac{d(m)^2}{|m|^s} = \frac{\zeta_A(s)^4}{\zeta_A(2s)} \ .$$

Use this to evaluate $\sum_{\deg m = n} d(m)^2$.